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ABSTRACT

The ideas and techniques involved in learning about fractions were investigated with students in grades 1-12, in the first 3 years of colleges, in community college mathematics courses, and in graduate school. Also included were some high school mathematics teachers, some mathematicians, and some retired persons. Part I provides the rationale and an overview of the project. Part II presents the theoretical background, plans for interviews, and the findings, with excerpts from tape-recorded interviews. An appendix contains transcripts of the interviews. It was concluded that nearly all students have access to good processing capabilities when dealing with concrete materials, and a major task of school programs is to build on these capabilities. The importance of good mental representations is stressed. (MNS)

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March, 1983

The Development of the Concept of "Fraction" from Grade Two Through Grade Twelve

Robert B. Davis

FINAL REPORT OF

NIE G-80-0098

PART ONE

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The Development of the Concept of

"Fraction" from Grade Two Through Grade Twelve

Robert B. Davis

This is Part I of the Final Report of NIE G-80-0098. Part II is bound separately.

I. This project began operation on August 21, 1980, and concluded on December 31, 1982. The chief investigator has been Robert B. Davis, assisted by Stephen Young, Patrick McLoughlin, and Carol Erb.

II. Purpose of the Investigation

We quote from appropriate sections of the proposal:

> The investigation proposed here has both a practical and a theoretical justification.
>
> Until recently it has been customary to design the mathematics curriculum on the presumption that concepts develop in a student's mind almost entirely on the basis of what we "teach" the student (quite often, on the basis of what we tell the student). Indeed, it was frequently assumed that the student's idea was a xerox copy of the teacher's idea. Recent research, especially in a neo-Piagetian tradition, has created a quite different presumption: concepts in the minds of students are not made by teachers, nor by texts, but on the contrary are made by the students themselves, primarily by revising earlier forms of the concepts. In this process, students take account of new inputs from teachers and texts, but do not use them as the sole basis for the construction of the new form of a concept. We shall refer to this new presumption as the developmental view of how a concept grows in a student's mind. [Cf., e.g., Davis, Jockusch, and McKnight, 1978; Andersson, 1976.]
>
> Taking this developmental point of view, we propose to study the ideas and techniques involved in fractions, as these ideas and techniques evolve in the minds of students, beginning in grade 2 (7-year-olds) and extending through grade 12.
>
> The practical reason for undertaking this study is a powerful one: fractions represent one of the two or three major obstacles to student advancement through the school mathematics curriculum. (The use of variables is the other largest obstacle; there may not be any third topic which represents an equally severe impediment to student progress.) Consequently, anything which smooths the way over the severe hurdles presented by fractions will improve the long-term mathematical performance of nearly all students, including especially those who are the most vulnerable to obstacles.
>
> We consequently argue that improving a student's progress toward mastering fractions should make a MAJOR contribution toward helping that student to get ahead in mathematics.

There is also a theoretical reason for undertaking this study. The proposers have been working for several years on various aspects of learning and "understanding" mathematics. An important theoretical question has emerged: mathematical performance is often cast in algorithmic or procedural form: "do this, then do this, then do this,...". But mathematical "understanding" includes also matters of visualization, meaningful interpretation, even everyday experience. There is, for example, the algorithmic procedure

$$\begin{array}{r} 74 \\ -25 \\ \hline \end{array} \qquad \begin{array}{r} {}^{6}\cancel{7}4 \\ -25 \\ \hline \end{array} \qquad \begin{array}{r} {}^{6}\cancel{7}4 \\ -25 \\ \hline 49 \end{array}$$

But this algorithm is clearly related to the idea that if you have 7 ten-dollar bills, you can (in principle) exchange one of them for ten one-dollar bills. How, precisely, is this "common sense" idea related to the subtraction algorithms?

We have become very interested in theoretical questions related to the interplay of conceptual understanding and algorithmic procedures. (We discuss this further, below, after preparing the necessary theoretical framework.) We do, however, anticipate by one remark: for surprisingly many children, ideas such as "changing one ten-dollar bill for ten one-dollar bills" are entirely unrelated for the subtraction algorithm, and do NOT help to improve performance in subtracting. This is quite unexpected, since most mathematically-sophisticated adults believe that the "changing the ten" kind of idea is very helpful to them in remembering the algorithm and in carrying it out correctly.)

Thus, in summary, the practical reason for undertaking this study is the prospect of finding how the ideas related to fractions develop and mature in a child's mind, as the child progresses from grade two through grade twelve. From a modern "developmental" point of view, we cannot determine this development arbitrarily, by merely choosing the content of the textbooks and the curriculum however we wish. On the contrary, unless the learning sequence bears a suitable relationship to natural developmental possibilities, effective learning is unlikely to result. Hence, knowledge of the type we are seeking is essential for the design of a more effective curriculum development of fractions, and a more effective mastery of fractions can make a very great difference in the mathematical careers of most students.

The theoretical reason for undertaking this study is the goal of obtaining more data on how "concepts" (such as "changing a ten-dollar bill") influence (or fail to influence) performance in carrying through an algorithmic procedure (such as "subtraction with borrowing"). Or, more generally, what can we learn concerning the relation between "ideas" (such as visual imagery) about the meanings of symbols and the manipulation of those symbols by rote algorithmic procedures?

The logic of our argument is, we believe, compelling, and goes essentially like this:

1. Examination of texts and tests shows that a mastery of fractions is essential to progress through the mathematics curriculum. [In addition to our previous discussion, see also Bates, 1978.]

2. Nor is this artificial: fractions play an essential role in many careers. (Cf., e.g., nursing. Here is a typical problem: A doctor orders 1/450 grains of a drug for Patient X. The nurse has available a vial labeled 1/300 grains per 2 cc., and a measuring device that measures in minims. A conversion table indicates that 1 cc. corresponds to 15 minims. How many minims should the nurse administer? [Saunders, 1980; Garcia, Note 2] Among adults interviewed by Saunders, 88% reported that the use of fractions was essential to the daily work in their job [Saunders, 1980]. By comparison, less

than half reported using metric measure, angle measurement, volume computations, square roots, or exponents. Fewer than 25% reported using trigonometric functions, logarithms, non-decimal numerals, or mathematical logic. Fewer than one in five reported using simultaneous equations or geometric congruence or set theory, and fewer than 10% reported using polynomial factoring, quadratic equations, or permutations and combinations. Saunders' survey looked at 100 different occupations.) Fractions ARE important.

3. Many people find fractions to be the most severe obstacle they face in their attempts to learn mathematics. This is the conventional wisdom of many experienced teachers, and it is also supported by objective data. For example, the testing program of NAEP (the National Assessment of Educational Progress) reveals results such as the following: whereas most 17-year-olds tested had mastered the arithmetic of whole numbers (e.g., 92% could correctly add a column of four 2-digit numbers; the same per cent could correctly subtract with "borrowing"; 76% could multiply 671 by 402 correctly), the results for fractions were far less encouraging: only 37% could estimate the sum 12/13 + 7/8 to the nearest whole number (i.e., 2), and only 39% could give a correct value for 250 divided by .5. NAEP summarizes their results by writing: "Assessment results indicate that teenagers do not understand the concepts of fractions, decimals, and percents." [NAEP, 1979, p. 2]

4. Mathematics is often the decisive obstacle limiting the career possibilities of many individuals. For example, among matters studied thusfar, mathematical difficulties are probably the main obstacle to the admission of women into medical schools; mathematical difficulties are probably the most serious obstacle preventing the enrollment of more blacks in engineering school. [Sells, 1973; Ernest, 1976; Sloan Foundation, Note 3; Treisman, Note 4.]

5. A more effective curriculum, to enable students to have more success in learning fractions, can only be built on the foundation of a better "developmental" understanding of how the relevant skills and concepts are created, by the students themselves, in the students' own minds. This is not a simple empirical fact that can easily be supported by experiment, survey, or citation. Rather, it is one of the fundamental precepts of a major modern school of thought in regard to teaching and learning. (We have referred to this, above, as the "neo-Piagetian" school, although it is considerably broader than the small "strict" Piagetian school. [Cf., e.g., Davis, 1971-2 A: Davis, 1971-2 B].)

Summary: But if math is important for jobs, and mastery of fractions is important for success in math, then a better curriculum in fractions is, necessarily, important. We believe that it is, and that this is the way to go about creating it. Before we can improve the curriculum, we need a deeper "developmental" understanding of the relevant learning. This study seeks to achieve such an understanding.

III. Method

We again quote from the proposal:

1. We start with a "zero-th approximation" outline of expectations. This tells us approximately what to look for. However, the inquiry itself will modify this substantially.
2. Students will be selected from classes in grades 2,3,...,12. The precise

number of students will be determined as the investigation proceeds. (This is the usual method in interview studies; it is impossible to predict beforehand how many students must be interviewed in order to refine the interview formats themselves. Furthermore, in interview studies one cannot predict beforehand what one will discover: who would have predicted beforehand that very young children would claim there are "more pennies" when a row of five pennies is spread further apart, or that kindergarten children would look at a bouquet of 8 tulips and 4 roses and claim that there were "more tulips than flowers," while simultaneously claiming that both tulips and roses are indeed flowers? Yet both of these are classical Piagetian results!)

The rule is to use enough preliminary students in order to develop effective interview protocols, then enough students to find the most common patterns of answers.

3. A variety of task-based interviews will be designed (in preliminary trials), and then employed (in final trials). Roughly, one interview pattern is required for each two or three ideas/skills that one wishes to explore. This will probably require one or two separate interview tasks for the lower grades, and perhaps as many as half a dozen for each of the highest grades.

The goal of the study, in any event, is to sketch in a broad map of the over-all territory, probably without reaching the finer levels of detail, which are properly left for subsequent studies. One must first have the broad picture; the details can only be added after that.

4. How is a task-based interview created? First one picks a general topic -- the distinction, let's say, between the "general" use of "one-half" to mean "some part of," vs. the mathematical use of "one half" to name a piece of a very specific size. [The general use of "one half" is analogous to the general use of "I'll be there in a minute," which does NOT refer to a precise 60 seconds, but really means "I'll be there fairly soon, " or words to that effect.]

Second, one invents one or more tasks that will bring into play the ideas or skills that one has chosen to study. [In the present example, one might ask a child to "take half of those marbles"; or one might begin by asking the child to read a story in which Mary asks Jill to give her half of something, after which one could ask the child questions about what Jill should do, what Mary meant, etc.]

In preliminary trials, one tests various interview tasks to see which ones work well with children of the age in question.

The interviewer sets the initial task for the child being interviewed, observes as the child works on the task, and then also intervenes in certain ways.

The interventions, planned and tested during the preliminary trials, can take several different forms:

(i) Sometimes no interventions are made, because one wishes to avoid influencing the child and possibily distorting the ideas that the child is about to reveal;

(ii) Sometimes questions are asked, such as "How can Mary tell whether Jill really gave her half, or not?"

(iii) Other types of questions include (depending upon the task and the age of the child):

(a) "How did you decide that?"

(b) "How do you know that?"

(c) specific content questions, such as:

"If I multiply $\frac{x^2 - 1}{x + 2}$ by x + 2, what answer will I get?"
[If the student immediately answers "$x^2 - 1$," one infers that the student knows the rule: $\frac{A}{B} \cdot B = A$; if, on the contrary, the student takes time to write

$$\frac{x^2 - 1}{x + 2} \cdot \frac{x + 2}{1} = \frac{(x^2 - 1)(x + 2)}{x + 2},$$

and so on, one infers that the student does NOT know this rule.]

(d) questions of the form: "How would you explain that to your younger brother John?"

(iv) Other types of interventions are sometimes useful, such as posing a modified version of the original task, etc.

5. Proceeding in this way, the original (hypothetical, or "zero-th approximation") outline of the development of the idea of fractions will be modified until gradually true developmental sequences are revealed.

Task-based interviews are recorded on audio-tape (in most cases) or on video tape (for a few interviews). In addition, both interviewers and one or more observers take notes, and the written work of the students is preserved. We have been using this technique for eight years, in a number of studies. While minor improvements are made from time to time, it is mainly a stable and reliable technique, in which we have considerable experience.

IV. Relation to a Postulated Theory

The proposal indicates also the basic conceptualization (or "point of view") to be employed. This is described, in detail, in Park II of this Final Report.

V. Part of a Continuing Program

This project is part of a continuing program of studying cognitive aspects of mathematical thought, that began at least as early as 1969, as a collaboration (at that time) involving Herbert Ginsburg (then at Cornell University) and Robert B. Davis (then at Syracuse University). From 1971 until 1976 this work was supported, in part, by the National Science Foundation, under grants GS-6331 (to Syracuse University) and NSF PES 741 2567 (to the University of Illinois). Contributors to this research have included, besides Davis and Ginsburg, Sharon Dugdale and David Kibbey (both specialists in computer-assisted instruction), Katie Reynolds Hannibal, Curtis McKnight, Jody Douglas (an anthropologist), Rose Grossman, Jack Easley, Stanley Erlwanger, Stephen Young, Patrick McLoughlin, Uri Leron, Schlomo Vinner, Edwina Rissland, Herbert Lin, and Tamás Varga. We have also benefitted substantially from conversations with Robert Karplus, Leon Henkin, Herbert Simon, Seymour Papert, Diane Resek, Roger Schank, Kristina Hooper, Oliver Selfridge, Piet Human, Jan Nel, Claude Janvier, Louise Poirier, Maurice Bélanger, André Boileau, Marilyn Matz, John Seely Brown, Elizabeth Stage, John Clement, James J. Kaput, and others.

We emphasize the on-going nature of this research program because we believe that this is what is required. Independent, short studies are unable to pursue questions to appropriate levels of depth.

Reports on earlier work in this continuing sequence include:

Erlwanger, Stanley . "Benny's Conception of Rules and Answers in IPI Mathematics." The Journal of Children's Mathematical Behavior, vol. 1, no. 2 (Autumn 1973)pp. 7-26.

Erlwanger, Stanley "Case Studies of Children's Conceptions of Mathematics." The Journal of Children's Mathematical Behavior, vol. 1, no. 3 (Summer 1975) pp. 157-283.

Ginsburg, Herbert. "The Children's Mathematics Project: An Overview of the Cornell Component." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 7-31.

Davis, Robert B. "The Children's Mathematics Project: The Syracuse/Illinois Component." The Journal of Children's Mathematical Behavior, vol. 1 supplement 1 (Summer 1976) pp. 32-59.

Davis, Robert B. "Children's Spontaneous Mathematical Thought." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 60-84.

Davis, Robert B. and Sharon Dugdale. "The Use of Micro-Assessment in CAI Lesson Design." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 85-102.

Davis, Robert B. "An Economically-Feasible Approach to Mathematics for Gifted Children." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 103-158.

Hannibal, Katie Reynolds. "Observer Report of the Madison Project's Seventh Grade Class." The Journal of Children's Mathematical Behavior, vol. 1 supplement 1 (Summer 1976) pp. 159-175.

Davis, Robert B. "Mathematics for Gifted Children - The Ninth-Grade Program." The Journal of Children's Mathematical Behavior, vol.1, supplement 1 (Summer 1976) pp. 176-215.

Davis, Robert B. and Curtis McKnight. "Classroom Social Setting as a Limiting Factor on Curriculum Content." The Journal of Children's Mathematical Behavior, vol.1, supplement 1 (Summer 1976) pp. 216-228.

Davis, Robert B. and Jody Douglas. "Environment, Habit, Self-Concept, and Approach Pathology." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 229-270.

Davis, Robert B. and Curtis McKnight. "Conceptual, Heuristic, and S-Algorithmic Approaches in Mathematics Teaching." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 271-286.

Davis, Robert B. "Selecting Mini-Procedures: The Conceptualization of Errors in Thinking about Mathematics." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 287-290.

Davis, Robert B. and Curtis McKnight. "Naive Theories of Perception." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 291-314.

Davis, Robert B. and Rose Grossman. "A Piaget Task for Adults." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 315-319.

Davis, Robert B. "Two Mysteries Explained: The Paradigm Teaching Strategy, and 'Programmability' ." The Journal of Children's Mathematical Behavior, vol. 1, supplement 1 (Summer 1976) pp. 320-324.

Davis, Robert B. and Curtis McKnight. "Modeling the Processes of Mathematical Thinking." The Journal of Children's Mathematical Thinking, vol.2, no. 2 (Spring 1979) pp. 91-113.

Davis, Robert B., Curtis McKnight, Philip Parker, and Douglas Elrick. "Analysis of Student Answers to Signed Number Arithmetic Problems." The Journal of Children's Mathematical Behavior, vol. 2, no. 2 (Spring 1979) pp. 114-130.

Davis, Robert B. and Curtis McKnight. "The Influence of Semantic Content on Algorithmic Behavior." The Journal of Mathematical Behavior, vol.3, no. 1 (Autumn 1980) pp. 39-87.

Davis, Robert B. "The Postulation of Certain Specific, Explicit, Commonly-Shared Frames." The Journal of Mathematical Behavior, vol. 3, no. 1 (Autumn 1980) pp. 167-201.

For examples of student work, see:

Suzuki, Kazuko. "Solutions to Problems." The Journal of Children's Mathematical Behavior, vol. 2, no. 2 (Spring 1979) pp. 135-165.

Burkholder, Bill. "The Perpendicularity of Radii and Tangents." The Journal of Children's Mathematical Behavior, vol. 2, no. 2 (Spring 1979) pp. 167-169.

Parker, Philip. "Theorems." The Journal of Children's Mathematical Behavior, vol. 2, no. 2 (Spring 1979) pp. 171-181.

Kumar, Derek. "The Reflection Property of the Ellipse," The Journal of Children's Mathematical Behavior, vol. 2, no. 2 (Spring 1979) pp. 183-200.

Kumar, Derek. "Points Nearest the Origin." The Journal of Mathematical Behavior, vol. 3, no. 1 (Autumn 1980) pp. 204-207.

Parker, Philip. "A General Method for Finding Tangents." The Journal of Mathematical Behavior, vol. 3, no. 1 (Autumn 1980) pp. 208-209.

VI. Data Collection

As planned, data collection has been via task-based interviews, with students from grade two through grade 12. For comparison purposes, task-based interviews have also been conducted where the subjects were teachers, and also where the subjects were experienced adults from various occupations. Over the entire duration of this project, more than six hundred interviews have been conducted. For details on these interviews, and also for a description of how the protocols are analyzed, see Part II of this Final Report.

VII. Results

The results themselves are presented in Part II of this Final Report.

VIII. Presentation of the Results

"Using Computers in Education." Invited paper presented at a conference in San Diego, California, March 12, 1981, sponsored by the Carnegie Foundation.

"Research on Number: Response to Glasersfeld et al." Invited paper presented to the Interdisciplinary Research on Number Symposium on Counting Types, University of Georgia, Athens, Georgia, April 6-8, 1981.

"Cognitive Science and Mathematics Education." Paper presented to the Third Annual Meeting of the North American Section of the International Group Psychology of Mathematical Education. Minneapolis, MN, September 1012, 1981.

"Response to DeVault." In: ASCD (Math Education), 5-19-81.

"Representations and Judgments in Mathematical Thought." Paper presented at 1982 AERA meeting, March, 1982, New York City. Session 10.13.

"Frame-Based Knowledge of Mathematics: Infinite Series." Journal of Mathematical Behavior, vol. 3, no. 2 (Summer, 1982) pp. 99-120.

"The Internal Structure of Mathematical Thought." Chapter in book by Professor Shoichiro Machida, Saitama University, Japan (in press).

"The Acquisition of Mathematical Knowledge." Paper presented at Conference on Computers in Education, USDE November 20-24, 1982, University of Pittsburgh.

with Edward A. Silver. "Children's Mathematical Behavior." In: Encyclopedia of Educational Research. Rand McNally, 1982.

"The Use of Informatics in Educational Programs." Paper written for International Conference Innovazioni Tecnologiche E Educazione. February 24-26, 1983, Venice, Italy. (in press)

"Complex Mathematical Cognition." In: Herbert Ginsburg (ed.), The Development of Mathematical Thinking. New York: Academic Press, Inc. 1983.

"Learning Mathematics: The Cognitive Science Approach to Mathematics Education. London: Croom-Helm, 1983.

"Mental Representations and Problem-Solving Success." Paper presented at Annual Meeting of the AERA, Montreal, Canada, April 15, 1983. Session 53.19.

"Cognitive Theories and the Design of Computer Experiences." Paper presented at the American Association for the Advancement of Science, Annual Meeting, Detroit, Michigan, May 31, 1983.

IX. Recommendations for Future Work

We believe that the kind of project undertaken in this study should be continued, but on a somewhat larger scale. With two or three senior investigators very valuable dialogues become possible.

March, 1983

The Development of the Concept of "Fraction" from Grade Two Through Grade Twelve

Robert B. Davis

FINAL REPORT OF

NIE G-80-0098

PART TWO

Curriculum Laboratory, University of Illinois, Urbana/Champaign

1210 West Springfield Ave., Urbana, Illinois 61801

March 1983

LEARNING FRACTIONS: THE IMPORTANCE OF REPRESENTATIONS

Final Report of NIE Grant G-80-0098

Part II

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[Part I deals only with financial reports, personnel, and similar matters; it is not a pre-requisite for Part II.]

I.

A. From the point of view of mathematical content, this study has focused on learning, understanding, and using fractions. Fractions are clearly an important impediment to the smooth forward progress of most students. Anyone who doubts this can try the experiment of asking relatives or neighbors

i) to explain why

$$\frac{1}{3} \div \frac{1}{2}$$

is found by the "invert-and-multiply-rule

$[\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}]$;

ii) to make up a reasonable "real" problem which leads to the need to calculate $\frac{1}{3} \div \frac{1}{2}$.

If the relatives or neighbors are not engineers, scientists, or teachers, they probably can answer neither (i) nor (ii). [Notice that "taking one third of a cup of sugar and putting half of it in each of two bowls" does NOT lead to $\frac{1}{3} \div \frac{1}{2}$, but rather to $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$, as the amount going into each bowl.]

Yet when a student enters secondary school, he or she begins the study of perhaps seven years of mathematics nearly all of which is based on fractions. The quadratic formula is written as a fraction

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(and its usual derivation depends heavily upon fractions); all six trigonometric functions are defined as fractions, and some of the most important relationships among them depend upon fractions; "similar figures" in geometry depends, in effect, upon fractions; probability is defined as a fraction; the slope of a line is defined as a fraction; in calculus, the derivative is the limit of a fractional expression; the process of integration, and the proof of trigonometric identities, often require skillful manipulation of fractions; in applied mathematics, many physical laws involve fractions. <u>A very high percent of high school mathematics, and college freshman and sophomore mathematics, depends heavily upon fractions</u>.

Yet we know that our present instructional programs do not enjoy much success in teaching fractions.

The result has to impose an extra burden on many students, if they try to continue their studies of mathematics. Little else need be said to explain the selection of "fractions" as the content focus.

B. But further questions remain. If fractions are the <u>content</u> topic, what <u>aspect</u> of "fractions" is to be considered? Our studies of school curricula [Davis, 1983A] suggest that <u>using</u> fractions, especially in novel situations, is the crucial aspect of fraction knowledge for most students who attempt to go on to more advanced mathematics. Consequently, this study looked especially carefully at how well prepared students were for using fractions in various ways, and particularly in novel situations.

This was not taken to mean that <u>understanding</u> was unimportant. Quite the contrary, it is precisely when one attempts to <u>use</u> a piece of mathematics in a novel situation that the demands for understanding become the most severe. In such settings, rote algorithms will often not suffice.

C. A third question remains, but it is a less obvious question. What sort of underlying theory will be used in shaping the study and in interpreting the data? At the present time there are, broadly speaking, two main contenders: an "external" look at behavior, comparisons then being made between student populations or alternate learning opportunities (which we shall call the "traditional paradigm") and what Thompson [1982] has called "the Constructivist approach" (we also speak of it as the "alternate paradigm"). The alternate paradigm holds that an underlying theory must be created, largely by proclamation (literally, by postulates), in order for science to advance in an organized way. Chemistry, as an example, could not have advanced to its present form without the postulation of elements vs. compounds, of nucleus vs. orbiting electrons, of electron layers to explain the periodic table, and so on.

For the alternate paradigm -- which is the approach employed in this study -- data is most commonly obtained from task-based interviews [cf., e.g., Herbert Ginsburg, The Development of Mathematical Thinking, Academic Press, 1983], and this is the method used here. A student is selected on some criteria, and is asked to sit at a table and to work on some mathematical task, while one or more observers watch. One observer may play the role of "interviewer" and interact with the student, by posing tasks, offering hints, asking (or answering) questions, providing supplies (pen, graph paper, calculator, etc.), and so on. The session is typically recorded on tape, student written work is preserved, and observer notes may also be available.

The student's significant actions -- choices, errors, written equations, assumptions, verbal explanations, etc. -- are then interpreted, insofar as possible, in terms of the postulated underlying theory. Hence this theory assumes considerable importance. We turn to it next.

II. Outline of the Postulated Theory

A. The theory used has been created in part at the Curriculum Laboratory, but it draws especially heavily on the work of Herbert Simon, Jill Larkin, and their colleagues at Carnegie-Mellon University, on the work of Roger Schank and colleagues at Yale, and on the work of Seymour Papert, Marvin Minsky, Andrea diSessa, Robert Lawler, and their colleagues at the Artificial Intelligence Laboratory at M.I.T. That it owes debts to Jean Piaget and George Miller should also be clear. For an extended treatment, see Davis, 1983-A.

At the beginning of this study the postulated theory could be sketched broadly as follows:

1. The statement of a mathematical problem, or some provocative situation or question, constitutes "raw input data."

2. Some feature or features of this raw input data triggers the retrieval, from the student's memory, of an abstract "problem representation," which, among other things, contains "slots" or "variables" into which facts from the input data are to be entered.

3. Facts from the raw input data are mapped into these "slots" in the problem representation. What results is the <u>instantiated problem representation</u>.

4. Checks are made to confirm that:

(i) the correct representation was retrieved from memory;

(ii) the mapping (in 3, above) has been carried out correctly.

5. Some features of the instantiated problem representation cue the retrieval of certain <u>representations of knowledge items</u>, some of which are "active" and can temporarily take control of the processing.

B. Certain aspects of this postulated mechanism require further comment:

1. <u>Tree search</u>. Much, obviously lies concealed beneath phrases such

as "some feature... of this raw input data triggers the retrieval, from the student's memory, of an abstract ... representation..." On any given day a large number of students will assert "I knew what to do; I just couldn't think of it during the exam!" Finding items in the human memory is no small task; on the contrary, we very often fail to locate items that do exist in our memories .. somewhere!

This memory search can be represented as a tree search of the usual form often seen in computer science, but with one major difference. In human behavior, we hardly ever possess (or construct) a complete tree (or a complete search space). There are nearly always many possible items which we neglect to consider. If all else fails -- and usually only then -- we may go back and "grow" some additional tree branches, corresponding to new possibilities that we had not noticed before.

2. In addition to the detailed "operational level" processing outlined in II A, above, there exists a "planning" or "control" level, that guides the information processing. (These levels may, or may not, be distinct.) How all of this works can be seen in examples, later in this report. [Cf. also Davis, 1983-A.]

3. The "phases" or "steps" outlined in IIA should not be thought of as large, unique entities appearing only once in a well-defined sequence. On the contrary, processing will typically cycle through the sequence many times, with quite a few cycles being incomplete.

A hypothetical example may make the point clearer. A student, confronted with the problem statement (= "raw input data")

Do the lines

$$x + 1 = 4t$$
$$y - 3 = t$$
$$z - 1 = 0$$

and

$$x + 13 = 12t$$
$$y - 1 = 6t$$
$$z - 2 = 3t$$

intersect?

might retrieve some representation forms learned in plane analytic geometry (such as

$$y = mx + b)$$

thus executing Step 2 in the IIA outline; Step 3 -- mapping input data into this representation -- might encounter difficulty (e.g. "There are so many different letters, x, y, z, t..."). Processing might then jump ahead to Step 4, which would fail to confirm a correct retrieval and correct mapping; processing might then cycle back to Step 1 (essentially, re-reading part of the problem statement), and this time the presence of "z" might trigger retrieval of some representation form from three-dimensional Cartesian geometry. The reader is left to develop further hypothetical possibilities -- for example, the word "intersect' might trigger the idea of simultaneous solution, hence (in Step 5) retrieval of knowledge items about simultaneous equations, which could lead the student to count, finding four variables x, y, z, and t (actually, this would be an incorrect -- but very likely -- count) and six equations. Many further, rapid, cycles through the sequence might lead to ideas such as

> number of equations ≠ number of variables: trouble
>
> maybe not
>
> which is larger?
>
> the number of equations

leading to an attempted simultaneous solution of the (incorrectly formulated) system

$$
\begin{aligned}
x \quad -4t &= {}^{-}1 \\
y \quad -t &= 3 \\
z \quad &= 1 \\
x \quad -12t &= {}^{-}13 \\
y \quad -6t &= 1 \\
z \quad -3t &= 2,
\end{aligned}
$$

apparently a system of 6 linear equations in the four unknowns x, y, z, and t.

If the student is resourceful enough, he or she may sooner or later realize that something is wrong; may come to develop a better picture of the role of "t"; may arrive at a correct determination of the actual number of unknowns (there are five); and may ultimately solve the problem...

...but it was <u>not</u> done

by a single pass through the five steps listed in IIA.

Cycling through the sequence can occur very quickly -- so quickly that no one, not even the solver himself or herself -- may be aware that that cycle occurred. Thus observation may miss many quick (and perhaps incomplete) cycles; but some cycles are slower or more dramatic or reveal themselves more clearly and <u>can</u> be detected. The actual number of cycles is probably larger than the number that will be observed -- possible <u>very</u> much larger -- but what <u>can</u> be observed is enough to be very helpful.

4. <u>It should be emphasized that nearly all of our studies use novel tasks</u>; <u>the student in our hypothetical example has probably NOT previously seen any problem of this type</u>. In part this is because our <u>teaching</u> makes very great use of novel tasks; on any typical task the student is probably attempting something for which he has NOT previously learned a "recipe."

C. Modifications of the Theory.

In Section IIA we said "At the beginning of this study the postulated theory could be sketched broadly as follows...". That phrase clearly suggests subsequent modifications, and these did occur. Four main alterations have emerged:

1. "Real-time" construction of representations. At the outset we focused our attention on the retrieval from memory of representations for the problem and also of representations for solution strategies. For example, a student seeing

$$x^2 - 20x + 96 = 0$$

(in, say, ninth grade) might retrieve from memory the problem representation

$$ax^2 + bx + c = 0,$$

might map input data into slots --

$$1 \rightarrow a$$
$$-20 \rightarrow b$$
$$96 \rightarrow c$$

-- and might retrieve the "solution" representation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Matching problem representation to solution representation then leads to the solution

either $x = 8$

or $x = 12$

[The "slots" in representation forms may, as here, be mathematical variables in the traditional sense, but they need not be, as (for

example) in the case of asking "Who is the First Lady in Britain?", and concluding that the "First Lady" in Britain is Margaret Thatcher's husband. For details, see Davis 1983A.]

Clearly, the retrieval from memory of an appropriate representation form does often occur. Indeed, from a cognitive point of view, this is probably an appropriate defining attribute for what is meant by a "routine problem" or "type problem" or "routine task."

But -- equally clearly -- this cannot be the whole story. We must often encounter raw input data that does NOT match <u>any</u> form previously stored in our memory. We therefore <u>cannot</u> retrieve an appropriate representation form.

We must, instead, synthesize a <u>new</u> representation. This will be assembled from bits and pieces that CAN be retrieved from memory, but will not itself be directly retrievable. [In subsequent experiences, of course, it may be retrieved.]

By the end of the study we had shifted our emphasis from the <u>retrieval</u> of representations to the <u>construction</u> of representations. This was an unsurprising consequence of our extensive use of novel tasks or novel problems, which could not be expected to match any previously-memorized representation forms.

Note that, to most lay audiences, this change would have moved us completely outside of the lay notion of "mathematics" -- for most people, mathematics <u>is</u> the use of a previously memorized recipe (i.e., algorithm). [Cf. Note 1.]

2. "Key-Feature" Representations. This important idea was developed by Stephen C. Young, at the Curriculum Laboratory. Young's principle might be stated as follows: <u>For nearly any mathematical or real entity, the memorized representation form is very sketchy and incomplete. It will typically include certain aspects, and will typically omit other</u>

aspects. Those aspects which ARE well-represented within some specific representation are called the key features within that representation. You may have several alternative representations, in your memory, corresponding to the same mathematical entity. The key features represented in one representation may be different from the key features represented in some other representation. Whether a given problem is easy for you to solve, or not, may depend upon which key feature representation you retrieve from memory, and how well its key features match the needs of this particular problem.

Young developed this concept in work with problems in geometry, but its applicability is universal. In his original work [cf. Young, 1982], Young dealt with problems involving pyramids. Most people develop a key-feature representation for a pyramid that resembles somewhat a tent -- a horizontal base, a vertical "tent pole", and a connecting-up of the edges of the base with the top of the tent pole. Using this representation, Young exhibits certain problems that are extremely difficult. With certain other representations, these problems become easy, because the key features of the representation make a better match with the needs of the problem. [for details, see Young, 1982.]

3. The Importance of Certain "Large Structures". This notion lies at the heart of a major controversy in education today: can "knowledge" be learned in very small, quasi-independent pieces? We do not deal with this controversy directly, although our experience strongly suggests -- at least to us! -- that one cannot learn mathematics as a large collection of small "facts" or procedures. The central essence of mathematics is in the intricate inter-connectedness of a very large number of pieces. Mathematics has a structure like a toccata and fugue by Bach. The miracle of it lies in how it fits together. One note, isolated by itself, tells us nothing.

While we do NOT deal with the general question of "small bit" decomposition of mathematics, our studies have compelled us to make more explicit a certain feature of the underlying theory: in mathematics one is often dealing with concepts any one of which, by itself, is necessarily represented by a very large representation structure.

To clarify what this means, we give some examples (including some from outside of mathematics, in fact), and also indicate some of the alternatives which this "postulate" excludes.

Example 1 (from outside of mathematics): A man may know the appearance of his wife's face. Could this be communicated to someone by small "bits" of some sort? Conceivably so; it might be possible to show someone, say, a one-square centimeter view of the wife's face, then another one-square centimeter view, until most or all of the face had been presented. There may be other ways to communicate the face via many separate, quasi-independent small bits. [Notice that a dot-matrix picture of the entire face does NOT qualify as a "small bit" communication, unless only a very few dots are exhibited at a time; if representations of large portions of the face are used, even though coded in dots, the "large structure" is being communicated directly. Conceivably such small-bit communications are possible. But, surely, it is far more effective to show a likeness of a large portion of the face, as in one or more ordinary photographs.

Example 2. (again, from outside of mathematics) Consider baseball. Suppose Person X knows absolutely nothing about the sport. Could a few small-bit "facts" about baseball give this person a good idea of the game? Clearly not. One must have a clear visual image of the layout of the playing field, the location of the players, typical postures and movements of players; one must know various kinds of pitches, the signals that are exchanged among players, a correct statement of the infield

fly rule, what home runs look and sound like, strategies that determine how to pitch to a given batter, the precise balance between how fast a ball can be thrown and how fast a player can run, and many similar things, if one is to be said to have a "reasonably accurate" mental representation of "baseball."

Example 3. (at last, an example from mathematics!) A <u>ring</u> is defined as a nonempty set R, closed under two binary operations, provided certain specific conditions are met. (The elements of R may, or may not, be numbers. We shall write the operations as + and $\cdot$, although these operations may, or may not, be "addition" and "multiplication" in their usual senses.)

The addition operation must satisfy:

$$\forall a, b \in R, \quad a + b = b + a$$

$$\forall a, b\ c \in R, \quad (a + b)+ c = a + (b + c)$$

$$\exists 0 \in R \ni \forall r \in R, \ r + 0 = r$$

$$\forall r \in R \exists -r \in R \ni \quad r + (-r) = 0$$

The multiplication operation must satisfy:

$$\forall a, b, c \in R, \quad a(bc) = (ab)c.$$

The two operations must be related by:

$$\forall a, b, c \in R, \quad a(b + c) = ab + ac$$

$$\forall a, b, c \in R, \quad (a + b)c = ac + bc.$$

If, in addition, [R, +, $\cdot$] satisfies the law

$$\forall a, b \in R, \quad ab = 0 \text{ implies that either } a = 0 \text{ or else } b = 0$$

and two additional laws,

$$\exists e \in R \ni \forall r \in R, \ re = er = r$$

$$\forall a, b \in R, \ ab = ba$$

then the ring [r, +, $\cdot$] is a more special kind of thing, known as an integral domain.

Now, what -- precisely -- <u>is</u> a "ring"? Can you give some examples of things that <u>are</u> rings? Can you give examples of some things that are rings, but are <u>not</u> integral domains? Do you have a clear idea of how integral domains differ from rings?

What point are we making?

The mental representation for a person's face must surely be a large structure of some sort. The mental representation of baseball must be a large and complex structure. The mental representation of "ring" (in its technical mathematical sense) must be a large and complex structure. Even the mental representation of "fractions" must be a large and complex structure.

We are NOT denying that these structures are probably built up, within a student's [illegible], in an <u>incremental</u> way. The first time I see a person, I do not successfully build a very complete mental representation for the appearance of that person's face. (If you asked me, the next day, what color eyes they had, I might not be at all certain.) Surely, many people enjoy baseball but could not correctly state the infield fly rule. As one watches many baseball games, one gradually learns more about baseball. Equally surely, few humans can learn all about rings in one single flash of experience. An adequate mental representation must be built up gradually. A student who knows that when a candy bar is shared equally between two people each person gets half knows a small piece of the concept of "fractions," but much more must be added, and an appropriate representation structure (or several structures) must be created in the student's mind.

Our point is that these concepts have an internal coherence that is an essential part of their nature. Seeing one square centimeter of someone's face is NOT "seeing their face." An adequate mental

representation <u>must represent this internal coherence</u>.

In particular, we take exception to those -- mainly non-mathematicians -- who say that "learning mathematics is a matter of learning vocabulary." The <u>name</u> for a concept is not the <u>concept</u> itself, no more than the ignition key that starts my car is the automobile itself.

As I write, there is a small four-legged animal that walks around on the floor. We have shared this house for ten years, so I know a lot about this animal -- its food preferences, its likely moves, its physical appearance in various postures, and so on. If you tell me the name of this individual animal, you have not thereby told me the <u>concept</u> of the animal. If you tell me the name of its species, you have not thereby told me the <u>concept</u> of its species. Both concepts are large, built up over years of experience. Both names are small. The <u>names</u> are a matter of vocabulary; the <u>concepts</u> are a matter of experience.

The <u>words</u> of chemistry, mathematics, and music are in the dictionary, but chemistry, mathematics, and music are not themselves in the dictionary.

We emphasize this seemingly obvious point because we find educational practice and theory are so often wrong in this regard, imagining that the word is the same as the concept. This is a common, and serious, error. Concepts are much larger, and must have far more elaborate mental representations. (Note 2.)

If one assumes that <u>building up a large representation for each key concept</u> is essential, the educational implications are numerous and important. In particular -- although we cannot prove it -- it appears that building up a large representation is an arduous task, which students often try to avoid. This avoidance often takes the form of a willingness to perform "small" (though possibly tedious) tasks, coupled with an unwillingness to think about the subject more deeply.

What does it mean to "think about the subject more deeply"? One

form this takes is the relating of input data to many previously-learned pieces of knowledge. We have reported elsewhere [Davis, Jockusch, and McKnight, 1978] on P., a seventh-grade student, who regularly made such observations, as for instance when he found that

$$y = f\ (x - 2)$$

shifted the graph of the function $y = f\ (x)$ two units to the right (when his first expectation was that subtraction should shift to the left); on this occasion he spent considerable effort reconciling this discrepant input data with some previously-learned expectations. By contrast, many students accept each "small" problem as an essentially independent experience, and make little effort to relate one experience to another. [for details, see Davis, Jockusch, and McKnight, 1978]. As a result, they do not build up large structures to represent important concepts.

Given our assumptions, in this study part of our attention was directed toward evidence for the building-up -- if it occurred -- of appropriate large representation structures to represent key concepts. [Cf., e.g., Davis, 1982]

4. Building on Simple Ideas. Finally, during the course of the study we have acquired a new appreciation for the important role played by very simple ideas -- ideas such as "this item is next to this item", "this item follows (comes after) this item", item A is equal to item B, putting things together, taking things apart, and so on. It seems accurate to say that a remarkably large part of even sophisticated mathematics is built up, in an elaborate way, from these very simple ideas, ideas which for the most part are already familiar to a 5 or 6 year old child.

III. Examples of Plans for Interviews.

In our use of task-based interviews, the interviewer's questions or other interventions are determined partly by the interviewer's *a priori* plans, partly by the student's answers, gestures, posture, mood (and so on), and partly by the interviewer's background knowledge. In our experience, a flexible format, highly adaptive to situational variations, reveals more than can be uncovered by more rigidly pre-determined procedures.

As a result of this approach, interviews cannot be predicted beforehand. In the main portion of this report, below, we shall see some transcriptions of actual interviews. These form our essential basic data.

By way of introduction, however, we sketch here a few pre-planned interviews. Note, however, that any actual interview would almost certainly depart from these *a priori* sketches.

A. Comparing a "real-world" context with a "smybolic" context. [Note: the student is presumed to be in grade 2, 3, 4, or 5; the interviewer always starts with "symbolic" questions in order to determine the student's level of competence in symbolic formulations. The reason for this is that nearly all students have a higher performance level in "real-world" contexts, and can often carry over this "real-world" competence into abstract contexts. Thus, if the interviewer starts with "real" tasks, the student may show a substantially higher level of performance in abstract questions posed later in the interview, but this may reflect new knowledge learned during the interview session (from the previous "real-world" tasks), and NOT reflect what the student had learned prior to the beginning of the interview session.

In order to deal effectively with this phenomemon, interviewers nearly always start with symbolic questions, locate student's approximate achievement level, then move to "real-world" tasks, then finally return to symbolic tasks to see if the student can, in fact, take the real-world knowledge just demonstrated, and use it to shape a notational or abstract calculational

procedure.

As a matter of fact, we construe this to be a desirable teaching strategy, although our present purpose is of course primarily <u>observation</u>, not teaching. (Apparently none of the schools where we have worked make much use of the strategy of solving real-world problems first, then using this as a basis for developing paper-and-pencil calculations. Cf., e.g., Davis and McKnight, 1980.)]

HYPOTHETICAL A PRIORI INTERVIEW PLAN:

Interviewer (henceforth, "I"): What have you been working on lately in arithmetic? What sort of problems are you doing?

Student (henceforth, "S"): Adding two numbers.

I.: Can you give me an example?

S.: How much is [writes]

$$\frac{2}{7} + \frac{1}{7} = ?$$

I.: How would you solve that problem?

S.: [writes]

$$\frac{2}{7} + \frac{1}{7} = \frac{3}{14}$$

I.: [having some idea of S's present level of understanding] Suppose you wanted to explain to a younger brother or sister what "one half" [writes: $\frac{1}{2}$] is. How would you do it?

S.: I'd say: "Suppose you had a candy bar and you divided it into two equal pieces. Each piece would be 'one half'."

I.: O.K. That's certainly right. What would 'one third' [writes: $\frac{1}{3}$] be? How would you explain that?

S.: Maybe you took a candy bar and divided it into three equal pieces.

I.: ...so, each piece would be 'one-third'. That's certainly right. You obviously understand that very well.

Which is larger, one-half [writes: $\frac{1}{2}$] or one-third [writes: $\frac{1}{3}$] ?

S.: One third.

I.: How did you decide?

S.: [some answer]

I.: Let's show it with pictures. Here is a candy bar, and here is another candy bar the same size. [Draws two congruent rectangles.] Would you divide the first bar into halves, and the other bar into thirds?

S.: [starts to draw, then stops and interrupts himself] Oh, one-half would be bigger!

I.: How did you decide?

S.: [now draws in halves and thirds, as he talks] Well, if you get half, you get a piece like this, but if you get a third, you get a piece like this [points to pieces].

I.: That's certainly right. Why do you suppose so many people give the wrong answer to that?

S.: Well... maybe they look at the 3 and think that must be bigger...

I.: [pursues this a bit further, then returns to main theme, with a question like]

How much is 'one half plus one half' [writes: $\frac{1}{2} + \frac{1}{2} =$]?

S.: One whole

I.: How do you write that?

S.: [writes: $\frac{1}{2} + \frac{1}{2} = 1$]

I.: O.K. Now how much is 'one third plus one third' [writes: $\frac{1}{3} + \frac{1}{3} =$]?

S.: two thirds

I.: Can you show me, with the candy bars?

S.: Well, here you have one third, and here you have one third, and if you put them together you have two thirds. So, one third and one third is two thirds. It's just like apples: one apple and one apple makes two apples.

I.: O.K.

Now, how much is 'one seventh plus one seventh [writes: $\frac{1}{7} + \frac{1}{7} =$]?

S.: Two sevenths.

I.: O.K. [I. wants to give a little more concrete meaning to "sevenths," which may be uncomfortably abstract for this student.]

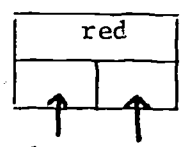
I have here some Cuisenaire rods. [Since the Cuisenaire rods may be somewhat unfamiliar to this student, I. at first uses the rods on familiar tasks. Consequently, he takes out a red rod -- actually 2 cm. long, but I. is careful NOT to mention this, which would switch the line of thought disastrously. I. puts the red rod down on the table.]

Can you find a rod which is <u>one-half</u> as long as this red rod?

S.: [puts white rod alongside the red rod]

I.: That's certainly right! Suppose somebody came along now, and said he didn't think the white rod <u>was</u> half as long as the red rod. What could you do to convince him?

S.: [puts two white rods alongside the red rod. They just fit.



]

I.: That certainly ought to do it.

Now, suppose I put down a <u>blue</u> rod. Can you find a rod that is <u>one-third</u> as long as that blue rod?

S.: [puts purple rod (4 cm. long) next to the blue rod.]

I.: Is that right? Suppose somebody doubted it; how could you convince them?

S.: [puts three purple rods alongside the blue rod (which is actually 9 cm. long). They don't fit.]

Uh-oh! That isn't right. Let's see... [sorts through the box of rods] I'll try this. [puts three light-green rods alongside the blue rod; they do fit.] O.K. The green rod.

I.: The light green rod is one third as long as the blue rod. It surely is --

you've just proved it!

[I. now switches to a different interpretation.]

O.K. Now I'd like to give names to these rods. If I call the red rod "one", what should I call the other rods? [I. and S. now work out names: white should be called "one half"; light green should be called "one and one half" or "three halves"; purple should be called "two"; and so on.

Then a new naming system is proposed: the light green rod is called "one". I. and S. work out names for the other rods: white is "one third"; red is "two thirds"; purple is "four thirds" or "one and one third"; dark green is "two"; blue is "three"; and so on.

I. now believes that S. sees how this "naming game" works. I. is now ready to home in on the question he really wants to explore -- S's earlier incorrect addition of $\frac{2}{7} + \frac{1}{7} =$.]

I.: O.K. Suppose I call the black rod "one". Can you find names for the other rods? [The black rod is actually 7 cm. long, but it is important NOT to say this to students; <u>if we "measure" in centimeters, there are no fractional values for rods.</u> In order to get sevenths, we <u>must</u> use the <u>black rod itself</u> as the unit of measurement. Metric lengths are mentioned here <u>only for the sake of readers who may not be familiar with the rods</u>.

S.: [works out that the white rod is "one seventh", the red rod "two sevenths", the light green rod "three sevenths", the purple rod "four sevenths", etc. The brown rod is "one and one seventh", the blue rod "one and two sevenths", and the orange rod "one and three sevenths".]

I.: Can you show me "one seventh plus one seventh"? [writes: $\frac{1}{7} + \frac{1}{7} =$.]

[Note that I. starts with a question he believes S. may answer correctly.]

S.: [puts down two white rods, alongside the black rod]

I.: Very nice! Exactly what I would do myself. Can you write that?

S.: [writes:

$$\frac{1}{7} + \frac{1}{7} = \frac{2}{7} \quad .]$$

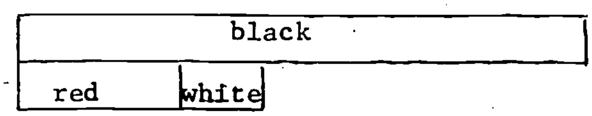
I.: Certainly right!

O.K. Can you show me "two sevenths plus one seventh"?

[writes:

$$\frac{2}{7} + \frac{1}{7} \; = \;]$$

S.: [puts down, alongside the black rod, one red rod and one white rod:



.]

I.: What single rod is as long as that red rod and that white rod together?

S.: The green rod [picks up a light green rod].

I.: The light green rod. O.K. And what do we call the light green rod?

S.: Three sevenths

I.: Can you write that?

S.: [writes: $\frac{2}{7} + \frac{1}{7} = \frac{3}{7}$.]

I.: O.K. That certainly seems to be right.

All right. I'd like to ask you a question. A few minutes ago [leafs back to student's earlier work] you wrote [shows work]

$$\frac{2}{7} + \frac{1}{7} = \frac{3}{14} \quad .$$

Now you write

$$\frac{2}{7} + \frac{1}{7} = \frac{3}{7} \quad .$$

Are those both correct?

S.: [At this point student responses vary so much that no "typical" response can be projected in advance. Most responses fall into one of the following four categories:

1. Yeah. <u>That's</u> [pointing to $\frac{3}{7}$] what you get when you do it with rods, and <u>that's</u> [pointing to $\frac{3}{14}$] what you get when you work it out on paper.

2. Oh! I made a mistake! It should be three sevenths. You're not supposed to add the denominators!

3. I'm pretty sure I did that right [pointing to $\frac{2}{7} + \frac{1}{7} = \frac{3}{14}$]; two and one is three; seven and seven is fourteen; yeah, that's [$\frac{3}{14}$] right! I don't understand this method of using those rods!

4. I... I'm not sure. [long silence] I don't know...

B. Planning, or Working Out the Story Line.

If many students fail to see mathematics as an abstract re-statement of "real-world" truths, this is not the only failing of school curricula. Nearly every effective use of mathematics has a "story line": an end goal to be reached if possible, and a sequence of steps or sub-goals that are intended to arrive, ultimately, at the desired final goal.

All students recognize this in stories; if Mary runs upstairs to get some money, students assume Mary has some reason for wanting this money -- perhaps, say, she has heard the ice cream truck coming along her street.

Yet within mathematics the idea that there should be a "story line" is hardly suspected by most students.

We designed some interviews to explore this territory (Note 3):

<u>SAMPLE HYPOTHETICAL INTERVIEW PLAN</u>

[The student is an 11th or 12th grader, in a high-school calculus course.]

I.: How would you deal with this problem?

[writes: $$\lim_{x \to 0} \frac{\sin 5x}{\sin 3x}$$]

[Note: the student has previously learned that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad .].$$

[Possible strategies include re-writing, as

$$\frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}}$$

$$\frac{5 \cdot \frac{\sin 5x}{5x}}{3 \cdot \frac{\sin 3x}{3x}}$$

$$\frac{5}{3} \; \frac{\frac{\sin 5x}{5x}}{\frac{\sin 3x}{3x}}$$

whence the limit is $\frac{5}{3}$.

C. Mapping and Retrieval

[Many problems require the student to retrieve from memory an appropriate representation form, and to map the raw input data correctly into the "slots" in the retrieval form.]

Sample problem:

Find $\lim_{h \to 0} \log \left(1 + \frac{h}{x}\right)^{1/h}$.

[The student, in a high-school calculus course, is presumed to know already the result that

$$\lim_{h \to 0} (1 + h)^{1/h} = e \doteq 2.718.]$$

The very common error is to allow oneself to be misled by superficial notational resemblances, and to map the input data into the retrieved form as follows:

input data		retrieved form
$\lim_{h \to 0} \log \left(1 + \frac{h}{x}\right)^{1/h}$		$\lim_{h \to 0} (1 + h)^{1/h} = e$
1	$\longrightarrow$	1
h	$\longrightarrow$	h
x	$\longrightarrow$	What? No image for x can be found.

This error is easily avoided if the student has developed an appropriate "meta" analysis:

Sample meta-analysis:

$$\lim_{h \to 0} (1 + h)^{1/h}$$

says "one, plus something small, with an exponent that is the <u>reciprocal</u> of the 'something small' ".

Now,

$$\lim_{h \to 0} (1 + \frac{h}{x})^{1/h}$$

<u>also</u> says "one, plus something small" ...BUT in this case the exponent is NOT the desired reciprocal of the "something small".

Hence, a sub-goal must be to MAKE the exponent become the desired reciprocal.

This is easily done:

$$\log (1 + \frac{h}{x})^{1/h}$$

$$= \frac{x}{x} \log (1 + \frac{h}{x})^{1/h}$$

$$= \frac{1}{x} \log [(1 + \frac{h}{x})^{1/h}]^{x}$$

$$= \frac{1}{x} \log (1 + \frac{h}{x})^{x/h} ,$$

after which the solution is easy.

[Notice that this also requires retrieval of the formula

$$K \log A = \log A^K$$

and a few other formulas about fractions and about exponents.]

D. Clarity of Concepts

The subject of fractions involves a number of concepts that may, or may not, be clearly understood. To explore this territory, we used interviews such as the following:

[The student may be anyone who has completed the fourth or fifth grade. Our study used this task also with a number of adults.]

I.: How would you explain 'one half' to someone?

[This nearly always elicited the "divide something into two equal parts" response, after which the interviewer went on to 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, and so on. These were usually defined as:

$$\frac{a}{b} = a \times \frac{1}{b} ,$$

a very reasonable "concrete" definition. "Divide the unit into <u>b</u> parts, and take <u>a</u> of them."]

I.: What is the answer to

$$4 \div 7 \quad ?$$

[This usually led, sooner or later, to the answer "four sevenths", written as

$$\frac{4}{7} \quad .]$$

I.: If I see the symbol

$$\frac{4}{7} \quad ,$$

how will I know whether it means "four divided by seven", or whether it means "take a unit, divide it into seven equal parts, then take 4 of them"?

[In fact, few students and few adults could answer this question.]

E. The Development of a Concept.

It is not always recognized that a person's concept of any topic, X, in mathematics will ordinarily have to change over time. This is clearly true <u>outside</u> of mathematics -- when I was in school, the "United States" was a nation of 48 states, and had never had a Catholic president. It is now a nation of 50 states, and it has had a Catholic president. In this case, history requires that my concept of the "United States" change.

Within mathematics, even when no changes are imposed from external reality, a student's concept of item X must change as the student learns more about mathematics.

One example is the meaning of the equals sign, = . For a young child,

$$2 + 3 =$$

means an implied action: take two things, take three other things, put them together, count how many you have. This young child, quite happy about

$$2 + 3 = 5,$$

will typically refuse to accept the legitimacy of

$$5 =$$

as a question, or the legitimacy of

$$5 = 5$$

$$5 = 2 + 3$$

(etc.)

as "answers". [The young child's view of = is very similar to its meaning on most hand-held calculators, which also accept 2 + 3 = 5, but will not produce 5 = 2 + 3.]

At a later age, after more mathematical experience, a child will accept

$$5 = 5$$

$$2 + 3 = 5$$

$$5 = 2 + 3$$

$$2 + 3 = 2 + 3$$

$$2 + 3 = 3 + 2,$$

and many more, because he interprets A = B to mean "A <u>is</u> <u>as</u> <u>many</u> <u>as</u> B." If he or she continues in mathematics long enough, they may come to a quite different interpretation:

> "A = B" means "A names some entity, and B is a name for this <u>same</u> entity."

In the case of fractions, this developmental evolution of each key concept is also important. At first, one probably defines $\frac{a}{b}$ by taking a candy bar, or a pizza, or something else, dividing it into <u>b</u> pieces, and taking <u>a</u> of them. This seems to be a nearly universal first meaning, and is probably an excellent starting point.

When one encounters "improper fractions," this meaning fails. One cannot, using this meaning, speak of (say)

$$\frac{5}{4} .$$

In order to deal with improper fractions, one must modify the definition of fraction. We must introduce the concept of <u>unit</u>. We divide each <u>unit</u> into <u>b</u> equal pieces, and take <u>a</u> of them. Now, with these new definitions, we can easily deal with

$$\frac{5}{4} ,$$

although we shall need two <u>units</u> in order to do it.

The study used some interviews intended to explore how well a student had succeeded in starting with one definition, then subsequently expanding this to provide for more powerful and more sophisticated definitions, which might not have been appropriate at earlier stages, but became necessary in later stages.

F. Technique in Calculations

Some interview tasks probed a student's skill in carrying out complicated calculations, as in

$$\frac{1}{\cot x} = \frac{1}{\frac{\cos x}{\sin x}} = \frac{\sin x}{\cos x} = \tan x,$$

or in simplifying

$$\frac{\frac{1-x}{2+x} - \frac{1-x}{1+x}}{\frac{2-x}{1+x} - \frac{1-x}{2+x}}$$

IV. Procedures

A. Range of Students Observed

Students included in the study were in grades 1, 2, 3, 4, 5, 6, 7 , 8, 9, 10, 11, and 12, in the first three years of college (with majors in anthropology and nursing to electrical engineering, mathematics, and computer science), in community college mathematics courses, and in graduate school. Also included were some high school mathematics teachers, some mathematicians, and some retired persons from non-mathematical professions.)

In terms of ability levels, students ranged from academically-gifted students at University High School, to the full range or "ordinary" students in four public elementary schools.

B. Range of Topics.

Topics used in task-based interviews ranged from second-grade arithmetic to second-year calculus.

C. Number of Students and of Observations.

As in other similar studies, we do not attempt to report the number of students observed, nor the number of observations. We do not report these numbers because any system of counting that has been proposed has had obvious serious deficiencies. Reported numbers would seriously distort the reality. The reasons for this include the following:

1. We never deal only with observing students. To do so, we believe, would be exploitation. If a student helps us by allowing us to see how he or she deals with some mathematical problems, we try to reciprocate by helping them to understand the subject better. Any "teaching" or "clarification" is of course provided after the task-based interview is used for observation. Yet we do not refuse to "observe" during subsequent teaching. If -- as often happens -- a student reveals something important about how he or she thinks, we try to record this, and may include it in our study. A policeman is never "off duty," and is supposed to attempt to maintain law and order at all times. We observe whenever we see something striking.

2. Students often reveal their thoughts most clearly in informal settings, or at non-pre-arranged times. While we schedule regular task-based interviews, some of the most revealing behavior occurs at non-scheduled times.

3. We are engaged in exploring relatively uncharted territory, a Lewis and Clark expedition into areas of some students' cognitive development in mathematics. We are not engaged in hypothesis testing, nor are we engaged in studies of the relative frequency of specific phenomena.

The question we seek to answer is, essentially, what happens when someone learns mathematics? What phenomena occur?

We believe that finding important phenomena in this area is NOT aided by common statistical methodology, nor by any extensive use of counting occurrences.

Our position is far less extreme than it may seem. If one is seeking possible life on, say, Mars, one does not want to overlook a solitary small red patch in an otherwise yellow landscape, even though statistical sampling may not include that red patch. Common sense

tells us to see if, perhaps, it has other differences in addition to color. When you are trying to identify new phenomena, you need to look at every possible occurrence that seems different or unusual. By the time you have accumulated a few instances of one specific phenomenon, you reduce your interest in adding even more instances to your collection, and focus on other possibilities.

Nonetheless, we can give some rough estimates of the number of hours of observation. Since on every school day we have at least two hours of observation, and probably never more than ten hours, the study includes at least 360 hours of observed student behavior, and not more than 1,800. Our estimate is that the actual total is between 700 and 1,000 hours.

D. Method of Data Collection.

Wherever possible, a task-based interview is tape recorded, and both the interviewer's notes and the student's written work is preserved. This is nearly always possible in the case of pre-scheduled interviews. [A few are on video-tape; most are on audio tape.]

The length of an interview is not pre-determined. Interviews are usually planned for about 20 minutes (unless a student is known to be inclined to work well for a longer time), but actual termination is determined

by the course of the interview. No interview is continued after a point where a student seems fatigued. Interviews must also sometimes terminate because of school schedules.

Unscheduled spontaneous sessions are not usually tape recorded, and must be reconstructed from observer notes and student written work.

Whenever an interview appears to add nothing valuable to this study, it is set aside and not processed further. It would be considered valuable if it indicated a new phenomenon, not previously encountered, or if it helped to resolve a question that was still in doubt. Ultimately, work-load limitations determine how many tapes are transcribed, and how many analyses are completed. [It takes over 20 person-hours of effort to make a transcription and analysis of a typical 20 minute session, and sometimes takes much longer.]

V. Real Contexts vs. Symbolic Contexts

We now begin the main section of this report: looking at transcriptions of a few interviews. To simplify the task of following interviews, we present complete sessions in appendices, and discuss shorter excerpts within the body of the report.

As a first example, we consider the question

$$\frac{1}{2} + \frac{1}{3} = ?.$$

The student in this interview is a generally bright, resourceful 5th grade girl, in an interview in June, at the completion of grade 5. The interview, and the tape, leave no doubt that she is personable, thoughtful, and resourceful. Interestingly, they also leave no doubt that her considerable sophistication does NOT extend to mathematics. She has recently changed schools, and reports that in her previous school there was very little instruction in arithmetic. The entire interview shows a bright girl who clearly can learn mathematics, and who can think about the subject very well. She even enjoys it! But her elementary school education

has done very little to help her learn mathematics. She is about to enter sixth grade, and has yet to learn more than the simplest ideas about fractions.

We call the numbered entries "utterances", though this is not really satisfactory; some are questions asked in the interview; some are statements; some are not spoken items at all, but are taken from the interviewer's written notes, or from the student's written work. Some, even, are explanatory remarks added after the interview. Lacking a better word, however, we continue to call them "utterances." The numbers are taken from the complete transcription and analysis, which appears in Appendix A.

The Excerpt:

175. I. I want to show you what most people think is really a hard problem, namely,

[writes: $\frac{1}{2} + \frac{1}{3} =$]

one-half plus one-third ...

Do you know what that is?

One half plus one-third

176. H. ...N-N-No... No.

177. I. Why do you suppose people think this is a hard problem? You just did "one fifth plus two fifths is three fifths". Now what makes "one half plus one third" harder than "one fifth plus two fifths"?

178. H. Ahm... Because one-half is larger...?

179. I. ... That's true, ... [i.e., one half is larger than one third -- but this, of course, is not the source of the difficulty, as

$$\frac{3}{7} + \frac{1}{7}$$

demonstrates.]

180. I. Let's come back to that...

That's the problem I really care about... but let's go on, for a moment, to a different problem.

Suppose I had one half plus one fourth

[writes:

$$\frac{1}{2} + \frac{1}{4} \quad] ,$$

what would that be?

181. H. Ahmm... one third. ...Ah... Two thirds!

182. I. O.K. [non-judgmental inflection]

Do you want to write that?

[I. was quite surprised by H 's answer; he suspects something interesting -- and something that he doesn't understand! -- is going on here . Hence, he wants to slow down the action, and get as much data down on paper (and on tape) as possible.]

What color (pen) do you want?

183. H. Ah... blue

[I. gives her a blue pen.]

184. H. So... what do I write?

185. I. Whatever you think the answer is.

One half plus one fourth equals... whatever you think...

186. H. Hmm... [pause]

Maybe... [pause]

No... [pause]

Maybe... two... thirds... or ... [reflectively; she is uncertain...]

[pause]

187. I. Well, you put down [on paper] anything you think the answer is, and then we'll see if we can figure it out with the rods.

[188. Remark. H's tone of voice makes this transition even more dramatically apparent. From the start of the interview, through the first 170 or so utterances [as numbered here], H. has been happy, confident, sometimes thoughtful, always resourceful, occasionally wrong -- but not often. And when she has been wrong, she has always easily corrected her error.

But now, beginning with #180, her tone of voice suggests that she is lost. She seems totally adrift, unable to find anything to hang on to.

This is the first big transition in this interview session, a session that can be divided into three parts:

Part I. H. is dealing with matters for which she can create adequate mental representations; she handles everything with resourcefulness and confidence.

Part II. We move into "paper-and-pencil" arithmetic for which she is unprepared. (Although students her age are ordinarily expected to know this content, it is clear that H. does not.) Specifically, we encounter the problems

$$\frac{1}{2} + \frac{1}{3} =$$

and

$$\frac{1}{2} + \frac{1}{4} = .$$

H. is no longer resourceful. She becomes unable to investigate, to set sub-goals, etc.

Part III. [which will come later].

When the <u>same problems</u> from Part II were tackled, in Part III, as <u>concrete</u> questions about <u>concrete</u> materials, H. could bring to bear all the resourcefulness she had demonstrated in Part I. and <u>could easily solve these problems.</u>

It has long been suspected that school programs *could* be created that could build on students' "concrete" or "experiential" or "informal" knowledge, and use this as a foundation for building up a powerful student capability for dealing with "formal" mathematics.

This has been difficult to demonstrate at the level of school programs, in part because the creation of potentially effective programs is no small task. But in this interview, at the level of ONE student at ONE MOMENT in her life, the possibility emerges clearly. IF one builds on concrete experiential knowledge, H. deals with these problems creatively and powerfully.

189. H. I guess it's...

[pause]

I'm not sure...

[pause]

190. I. Well, let's try the rods...

If we want to talk about "one half" and "one fourth", which rod do you suppose we want to call *one*?

191. H. [pause]

Ahmm...

[pause]

Light green?

192. Silence]

193. I. That's a good guess...

...but I think that *might* not be our best bet...

194. [silence]

195. I. Remember, when we call the light green rod "one", we get *thirds*, OK?

196. H. Um-hm. [inflection indicates agreement]

197. I. ...I don't think you're going to get <u>fourths</u> or <u>halves</u> [if you call the light green rod "one"].

198. H. Oh, yes. [Her tone of voice indicates that her self-assurance and resourcefulness are rapidly returning.]

It's probably the magenta or purple. [The "purple" or "lavender" or "magenta" rod is 4 cm. long, and is the optimal choice for this problem.]

199. I. Yeah. Exactly right.

That's just what we should do.

Now... so we can keep track, I'm going to stand a purple rod on end, here, where we can both see it. That should remind us that the purple rod is "one".

O.K. If that purple rod is "one", which rod would you call "one half"?

200. H. The red

201. I. Exactly correct!

Yeah.

That's exactly right.

And now... which rod will I call "one fourth"?

202. H. The beige

203. I. The beige. Good. Exactly right.

O.K. So, what happens if you add one half and one fourth. What will you get?

204. H. Three fourths [Note: at this point, H. does not need to use the rods! They have served their purpose -- we would say they have enabled her to build up an appropriate representation in her mind; <u>she can now solve the problem easily just by thinking about it</u>. Because she now has an adequate mental representation, she is <u>able</u> to think about it!]

205. I. You're right!

So... let's not <u>erase</u> that answer [her previous answer,

$$\frac{1}{2} + \frac{1}{4} = \frac{2}{3}\ ],$$

because I think that's interesting, but here we have "one half plus one fourth"

[writes: $\frac{1}{2} + \frac{1}{4} = \_\_\_$,

below the preceding incorrect result],

and... what do you get now?

206. H. Three fourths

207. I. ...you want to write that answer...?

[208. She does.]

209. I. That's really a very nice job!

O.K.

Let's come back to the <u>really</u> hard problem.

I don't think it will be hard for you, but a lot of people find it <u>extremely</u> hard.

[writes: $\frac{1}{2} + \frac{1}{3} =$]

If you want to talk about "one half" and "one third" -- well I need to be able to find <u>one half</u> of some rod, and I need to be able to find <u>one third</u> of it...

...and so... <u>which</u> rod will we call "one"?

210. H. Ahmm... the dark green?

[Of course, she has chosen correctly!]

211. I. I <u>think</u> you're probably right. [Actually, I. <u>knows</u> she is right, but he doesn't want to remove the need to check it out.]

You really are very good with those.

Let's try it, and check up and see...

If dark green is "one," which rod is "one half"?

212. H. Maybe purple... No... Ahm... Green! Light green!

213. I. Light green! Exactly right!

So we know, then, that light green is "one half"...

... and which rod would you call "one third:?

214. H. [instant response] The red!

215. I. You're right!

O.K.

So what, now, will "one half plus one third" be?

216. H. Ah...

[pause]

...one third is...

I don't know...

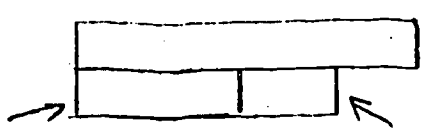
...four fifths?

217. I. Can you do something with the rods so that you can show me a rod that [illegible]ents the answer? We'll worry about what to call that rod later on... Let's find the rod, first...

Show me a rod that's the right size...

218. [pause]

[219. H. puts down a dark green rod, then a light green rod alongside it, then she adds on, end-to-end, a red rod:



light-green rod   red one]

220. I. So... how are you going to show me "one half plus one third"? Just exactly what you're doing!

Will you describe what you're done? [So as to get it on the audio tape...]

221. H. I just took the dark green and put the light green -- "one half" -- right next to it , and then the red, as a third... [pause]

222. I. That's exactly right!

[pause]

223. I. We need a name for these. Some people call these "trains", because they look a little like a train, with cars in a row...

So , you made a train of a light green and a red, right next to the dark green rod, there,...

... and that's certainly the right answer. That is the right size, to be "one half plus one third"

....and now we need to figure out what to call that...

[pause]

What do you suppose we should call it?

224. H. Ahmm...

I don't know...

225. I. [very softly] Well...try...

226. [pause]

227. I. How could you decide?

228. H. I'm trying to think of something... I just can't...

229. I. O.K.

Now you're picking up one of the rods. Which rod are you picking up? [I. wants to get a record on the tape.]

230. H. Ah... beige...

231. [pause]

232. H. ... I don't know...

233. H. ... probably six or seven [she is estimating how many beige (or white) rods would fit alongside the dark green rod]

234. I. O.K. What will we call the white rod, then? The beige one, that is?

235. H. One seventh. [Her estimate is wrong, but her method is entirely correct. If it were true that seven beige rods would make a train as long as one dark green rod, then it would also be true that the beige rod would represent one seventh. But, of course, her visual estimate is wrong. The fact is that six, not seven, white rods make a train as long as one dark green rod.]

236. I. [I. does NOT want H., or any student, to be looking to the teacher (or the interviewer) for the determination of "truth". The strength of mathematics and science is that they are NOT entirely authoritarian; one attempts to determine the truth directly, wherever possible.]

How would you settle it?

Let's do something to be sure...

237. [silence]

238. [H. actually lines up a train of white rods alongside the dark green rod.]

239. H. It's six! [Happy to have it settled.]

240. I. Exactly correct!

So... we call the beige rod...?

241. H. One sixth.

242. I. ...and what are you going to call the train that you made up? With the light green and the red?

243. H. Hmm... Three sixths?

[244. H.'s visual estimation is surprisingly poor... Many students do much better.]

245. I. You want to check up and make sure you got it right?

[246. silence]

247. H. Hmm... No... Five sixths!

[Again, she sounds pleased to get the matter settled.]

248. I. That's exactly right!

O.K.

Let me write that...

[He writes

$$\frac{1}{2} + \frac{1}{3} =$$

on a new line on the paper.] I'm writing "one half plus one third equals...", ... and will you write what you now think that is equal to?

249. H. writes:

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

250. I. That's really a nifty job!

That's very nice.

You're really very good at mathematics.

You say you just started studying fractions?

251. H. Yeah. I didn't have very much [about fractions] last year. I've just come to this school. [Before that] we didn't have very much fractions...

252. I. Where were you before you came here?

253. H. In Michigan. Near Pontiac and Detroit.

254. I. O.K. You've been a big help to me. [The students understand that these interviews are part of a research program.]

If everybody could deal with math the way you can, it'd be wonderful! [This was a sincere remark. Notice that H. has been able to fight her way through to a correct answer in each key problem! She did not come to this session already knowing how to

add fractions, but she has used her determination and her very considerable resourcefulness to work through the problems, and to arrive at correct answers.]

What do you suppose I should say to other people in order to help them to get to be better at mathematics?

255. H. [pause]... I don't know...

[This may partly be modesty...]

256. I. What I think _you_ do, that is really _terribly_ valuable, is that you really _think_ about the problem.

257. H. Oh... I wouldn't know what to _say_ to them... [i.e., to anyone who needed help] ... I wouldn't know what to _tell_ them...

258. I. It seems to me that when you are working on a math problem, you really think about it. A lot of other people want to be able to solve problems _without_ really _thinking_ about them... That doesn't usually work... You _have_ to think about it...

Do you have any questions _you_ want to ask _me_?

259. H. No... not really... I think I understand it.

260. I. O. K. You've been a big help to me.

So, we see that H. is well able to deal with

$$\frac{1}{2} + \frac{1}{3}$$

in the concrete context of Cuisenaire rods, but was NOT able to deal with it in the earlier part of the interview, when it was presented merely in terms of symbols written on paper. In this she is entirely representative of nearly all students whom we have studied. [She is unquestionably brighter than most!]

VI. "Noun" Knowledge vs. "Verb" Knowledge: The Synthesis of a New Representation Form.

In observations dating back even to earlier studies, spread over 10 years and hundreds of students [cf. Davis, Jockusch, and McKnight, 1978], we have consistently seen a pattern of students who can carry out some sequence of actions, but seem unable to talk about the sequence. We have previously called this having verb knowledge (they can do it), but not having noun knowledge (they cannot talk about it). The phenomenon should be familiar to everyone. We know children who can lead you by the hand to their "favorite hiding place" (or whatever), but could not stand there (in, say, the kitchen) and tell you how to get there.

In terms of our postulated theory, we would say: these students have not yet synthesized in their own minds an adequate representation for the process in question. Of course, later on -- after sufficient experience -- an appropriate representation form will have been synthesized.

In these terms, the next interview, with a first-grade boy, S., is a very exciting one. We see the representation for a process being created, as it were, in front of our very eyes!

The task is to find fractional parts of discrete collections, and to write the results as

$$\frac{1}{2} \text{ of } 8 = 4$$

$$\frac{1}{2} \text{ of } 6 = 3$$

and so on.

At first, S. is able to share out 8 things between two people, etc., but after he has done it he cannot hold the action in his mind and report correctly on what he did. We would say the necessary representation form is not available in his mind, so he cannot map the data of the experience into an appropriate representation.

Slightly later in the interview a suitable representation form -- that relates an initial amount, 2N, to "my share", N, and to "your share," N -- begins to take shape. It is shaky, and in use it is unreliable, but its operation can be discerned.

In particular, it is unreliable because input numbers are not necessarily mapped into the proper "slots" (i.e., the slot for "total number," the slot for "my share," and the slot for "your share"); we see instances of "mapping errors."

It is also unreliable because, not being automatic, it imposes a heavy cognitive load on S."s processing capabilities, and leads him to make errors in counting and in other concurrent processing tasks.

By the end of the interview, S.'s mental representation for "sharing equally" is becoming a little bit firmer, but is not yet reliable.

Interview Excerpt:

15. [I. puts 4 red rods on the table.]

16. I. Suppose we were going to share these 4 rods -- pretend they're candy or something. I was going to get half, and you were going to get half. How many would you take?

17. S. I would take two. [S. says this immediately and confidently, obviously pleased that he can answer so quickly.]

18. I. Yeah...that would be exactly right!

You didn't have to think very long to do that! You knew that right away!

19. S. ...cause I knew two plus two is four [again, happy and confident]

20. I. Unh-huh... and so...[pause in talking while I. gets some more red rods out of the box] Let's see...that's one, two, three, four,... seven... Is that eight? [i.e., "Do we now have 8 red rods on the table in front of us?"]

21. S. Unh-huh [intonation implies a confident "Yes"]

22. I. O.K. Suppose <u>now</u> you were going to take one-half...

How many, <u>now</u>, would you take?

23 S. Ahh... [pause of about 3 seconds]

Four

24. I. You're right. You're certainly right.

Why don't you <u>take</u> four,...and I'll take four...

[They do so.]

Now we can pile them up and see if we actually got the same number. We can see if your stack is the same height as my stack.

[They do so, and the stacks match.]

So... we each got <u>half</u>.

[pause; then sound of rods clinking against one another as I. searches through the box of rods, looking for some more red rods.]

You're very good at this, you know.

How many red ones have we got here, altogether?

25. S. Ten

26. I. How did you figure that out? [Stefan has not done any obvious direct counting of the rods.]

27. S. Well, because I knew that four plus four is eight, and [then] I counted "nine, ten" [with his eyes, not his fingers].

28. I. Perfect!

Suppose you were going to take half of all of them. How many would you take?

29. S. I would have ... five! [again sounding happy and confident]

30. I. You are very good at that! You're really very good!

Oh, wow!

...[pause of about one second]

Well, if you were sharing with three people, then you'd say you were taking thirds. Let's see if we have any more red rods in here [i.e., in the box; he finds two more]

How many red rods have we got now?

31. S. I don't know ... [pause]...Counting all these? [gestures to the various piles of red rods on the table]

32. I. All those...

33. S. Eight

34. I. I think it's more than that...

35. S. Every single one of them [i.e., of the red rods on the table]

36. I. Yeah ...

37. S. Twelve

38. I. Yeah, twelve...

How did you do it?

39. S. Well, first I counted eight from these... then I went "nine, ten, eleven, twelve."

40. I. Now, instead of just you and me, suppose we were going to share these among three people... Suppose you were going to get some,

and I was going to get some, and maybe we had some third person...
[I. decides he needs a more concrete piece of imagery, so he interrupts himself.]

Who in the class would be somebody else we might share with? Just give me a name...

41. S. Kelly

42. I. Kelly? All right. You're going to get some, and I'm going to get some, and Kelly is going to get some. We want to be fair, and all get the same number.

How many do you suppose we'd each get?

43. S. ...each get four! [answered very quickly, happily, and confidently.]

44. I. How did you figure that out?

45. S. Well, I knew that we'd each had four, and then I counted four... [Notice that this does NOT fully explain how S. thought about this problem. He is NOT referring to the immediately preceding problem, which dealt with one-half of ten being five. Possibly S. used some kind of "estimate, then count each (imagined) pile to see if the division was fair. He is NOT actually touching or moving the wooden rods, here. Indeed, probably neither he nor the Interviewer could have solved the problem so quickly if they had attempted to move the actual rods around on the table. S.'s answer came very quickly!]

46. I. You are very good at that!

People would often say, because there were three of us -- you, me, and Kelly -- because there were three of us, we each got one third.

And so we could say: "one third of twelve is four."

47. S. Un-huh [agrees; presumably he has never heard this language before, or at least never attended to it...]

48. I. Let me show you how you'd <u>write</u> some of those.

[He writes:

$$\frac{1}{2} \text{ of } 8 = \quad ,$$

and reads it aloud.]

I could say "one half" -- people write "one half" like this [as he writes the

$$\frac{1}{2} \quad]$$

...have you ever seen that before?

49. S. Yeah [apparently he has]

50. I. ...we could write "one-half of eight is..."

That's one we did just a minute ago...Do you remember what that is? How much is one-half of eight?

51. S. No.

[This is in some ways a surprising result. A few minutes earlier S. had worked this out, and was quite confident about it. We would analyze this as an instance of "verb" behavior -- something you can <u>do</u>, given a flow of externally-supplied feedback (such as seeing the actual rods), but have not yet transformed into <u>noun knowledge</u> -- the process is not yet a sequence of actions, welded together cognitively, so that you can <u>think about</u> the process <u>without</u> actually doing it.]

52. I. You want to think about it for a few minutes and see if you can figure it out? One half of eight? See if you can figure that out...

[pause of about 3 seconds]

53. I. What do you think?

54. S. No

55. I. Well, what could we... If we wanted to make that into a <u>problem</u>, ... [Note that S.'s attention is now mainly on the written notation $\frac{1}{2}$ of 8 =, on the paper, and not on the rods.]

What that says is, we had eight things, and you and I wanted to share them equally, so that we'd each get <u>one-half</u>.

How many would you get?

56. S. I'd get two. [Again, confident -- but this time he's wrong.]

57. I. Well, there are eight ... if we each take one half...

58. S. Oh! I get it! Six. [confident, but wrong]

59. I. Six [repeating S.'s answer with what I. hopes is a non-judgemental inflection]

60. S. Yeah! Four, five, six!

61. I. ...but we've got eight altogether. [S.'s attention, for utterances 48 through 61, has been on the written problem on the paper, and not on the rods.]

62. I. This number here is now many we've got altogether [indicates the '8' in

$$\frac{1}{2} \text{ of } 8 =].$$

63. S. Oh! Hmm... Ahhh...

64. I. If we share them equally, there are <u>two</u> of us, so we'd say we each get <u>one half</u>.

65. S. Seventeen

[At first, this choice seems to defy explanation. But wait!]

66. I. How did you figure that out?

67. S. Well... First I went 'eight', and then real quickly I counted 'nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen'.

[S. has taken the '8' as one person's share, and computed what the original total must have been ... but in his 'real quick' counting,

he has made an error, and gotten 17 when his method should have led to the answer "sixteen".]

68. I. ...Well... Let me come back to this later. [I. feels there is now too deep an accumulation of confusion surrounding this problem, so he chooses to go on to a fresh start on a new problem.]

Another one! Let's try some other numbers. [He also focuses his attention on the rods, rather than on the paper.]

Suppose we have...

How many of those red rods have I got there, now? [I.e., on the table.]

69. S. Six

70. I. O.K. Suppose we shared those fairly, so you got the same number I got.

How many would you get?

71. S. I'd get...three! [He actually starts to share out the red rods, but his imagination jumps ahead of his fingers, and he proclaims the answer, without completing the "sharing-out" distribution.]

72. I. Yeah. And people would say that you got half, and I got half, because there were <u>two</u> of us. [Note: In utterances 69-72, I. has revised the sequence, compared to utterances 48-67. In 48-67, the temporal sequence <u>starts</u> with a notation on paper, and hopes to elicit the construction of a playing-out of the sharing process with wooden rods; by contrast, the sequence 69-72 <u>starts</u> with a sharing of physical rods, and then uses records written on paper as a means of recording what was actually done.]

73. I. And so we could write that and say [writing as he talks] "One half of six is...--" let's see: the whole number was six, and you got half of that... How many did you get?

74. S. ...six? ...

75. I. That's right, the whole number of all of the rods we started with was six, and you got half of that... How many did you get?

76. S. ...ah...three [reasonably confident]

77. I. So... we say "One half of six is three!" [writing it as he says it]. Do you see how that works?

78. S. Yeah. [sounds reasonably happy about it]

79. I. Suppose we had four to start with [he assembles 4 red rods in a pile on the table] ... What would one half of four be?
[pause of 7 seconds]

80. S. Eight.

[Here he makes exactly the same error he made earlier (65-67) -- he has retrieved an immature frame that has three place-holders, that we might call A, A, and 2A. However, both now, and in utterances 65-67, he takes an input datum that should match with the 2A slot, and matches it with one of the other slots. In the present case, he consequently gets 4, 4, and 8. The next two utterances serve to clarify somewhat the method he is using.]

81. I. How did you do it?

82. S. Well... I knew that four plus four was eight! [triumphant!]

83. I. Oh... But suppose four is all there are, and you're going to take half, and I'm going to take half...

84. S. How many would I take away?

85. I. Yeah

86. S. Two [very confident, and pleased that he understands]

87. I. Yeah... and that's what we would say that was.
[writes: $\frac{1}{2}$ of 4 = 2]

[Comment: The Interviewer worked to show S. that his first frame-retrieval-and-mapping (utterances 80-82) had NOT succeeded, and I. tried to use the experience of "sharing out equal portions" to begin building a more mature frame in S.'s mind. At least in

the short run, I. seems to have been successful.]

88. I. [Reviewing what has just been done with rods, and recorded on paper...] Let's see; we've done... for half of six, we got three, and for half of four we got two...

Let's think about this "half of eight" problem [returning to the first problem on the paper, but now writing it over, to preserve the time-sequence order of the lines on the paper]

89. S. [sighs]

90. I. What <u>could</u> we do? To think about this "half of eight" we need to make up a story about sharing ... if we want to match "one half of eight," then how many red rods should we start with?

91. [pause of about 6 seconds]

92. I. Altogether we should have <u>eight</u>, right?

You count out enough to have eight.

[S. does so.]

Now, if you wanted to find out how much <u>half</u> of eight is, you could share them equally with me, and what <u>you'd</u> get would be <u>half</u>, and what <u>I'd</u> get would be <u>half</u>.

[I. is being worried that he has not yet made contact with a foundation of things that S. knows well, so I. interrupts himself to seek a firmer foundation in things that S. already knows...]

Do you have any brothers or sisters?

93. S. Yeah. I have one sister.

94. I. Have you ever shared a candy bar with her?

95. S. Yeah.

96. I. How do you do it?

97. S. Well... I split it... and then I give one half to her...

[pause]

98. I. ...and keep the other half for yourself...

99. I. O.K. Suppose we wanted to do that with these [the 8 red rods in a pile on the table]. These might be marbles, or something... and you are supposed to get <u>half</u> of them, and I am supposed to get <u>half</u> of them...

How many would you take? [pause for about 6 seconds]

100. S. Four

101. I. You're right!

Can you <u>write</u> that... say "one-half of eight would be..." How many?

102. S. Four

103. [I. writes

" $\frac{1}{2}$ of 8 = "

and S. finishes it, by writing "4".]

104. I. What do you suppose <u>one-half</u> of <u>twelve</u> would be? That's really a hard one...

105. S. That's a hard! [S. uses the adjective as a noun]

106. I. [agreeing] That's a hard one!

107. S. [sounding as if he's guessing] Eleven?

108. I. How did you do it?

109. S. Well...because...I knew that it was <u>one</u>, and then I added twelve more... [Clearly something has malfunctioned.] ...and I counted them...up...

110. I. I see ...

111. I. What'll we do? We wanted <u>twelve</u> to start with, right?

Can you write "one half"?

One half of twelve?

112. S. S. writes

$\frac{1}{2}$

113. I. "...of twelve..." See, if we each get half, then how many we get depends on how many we start with, right?

114. I. Maybe we can figure it out...

First of all we <u>need</u> 12 [I. counts out some more red rods]

O.K.

Now, how much is <u>half</u> of 12?

115. S. Eleven?

116. I. Well, what will we <u>do</u> with these pieces of wood?

117. S. Ahh...

118. I. If you and I took the <u>same amount</u>, then you'd get <u>half</u>... Isn't that right?

119. S. Yeah!

120. I. So...let's do that!

121. S. How many I would get?

122. I. Yeah...

123. S. Ungh! [or some such sound]

[pause of about 6 seconds]

124. S. I would get...Seven?

125. I. Let's do it, and see!

[They share out the red rods.]

126. S. [really triumphant] Six!

127. I. [agreeing] Six [This is plainly apparent, since the rods have been shared out.]

So... we can say "One half of twelve is..."

128. S. Six!

129. I. You're right!

That's exactly how it works!

Let's see... what have we done? We've done 4, and said half of four is two... We've done 4, and 6, and 8...

Oh! Ten! We haven't tried ten. What do you suppose half of ten would be? How many would you have, if you had half of ten?

130. I. Can you figure it out, before we do it?

131. S. Six?

132. I. Well... let's try it, and see...

133. S. I mean... ungh...

[They share out the ten red rods, equally.]

134. S. Five [entirely confident]

135. I. Five. You're right! Why don't we write that one?

One half of ten would be five.

[S. writes. Actually, in both of the last two equations he has omitted the word "of". I. sees possible trouble stemming from this, but decides this is not the time to bring the matter up.]

136. I. That's exactly right!

VII. Developmental Modification of Concepts.

We have sketched, earlier, how the (usually implicit) "whole" that we divide up in early work with fractions must be extended to the concept of unit in order to deal with rational numbers larger than one. Here is a relevant interview excerpt [this is the same student we saw in Section V; for the complete interview session, cf. Appendix A.]:

1. I. O.K....We're thinking about fractions . . .

2. H. Um-hmm [agreement]

3. I. Let's take a very easy one -- let's take <u>one</u> <u>half</u>

[writes $\frac{1}{2}$].

Suppose you wanted to <u>explain</u> "one-half" to somebody --

[interrupts himself, in order to establish a more concrete task setting:] Do you have any younger brothers or sisters?

4. H. I have a step-brother who's younger -- he's nine... <u>eight</u> or nine...

5. I. ...so he probably already knows about "one half"... but you can imagine a <u>fairly</u> young child who really didn't know about fractions?...

How would you explain "one half"? To him? Or to her?

6. H. Ah...Probably take some...something that can be <u>divided</u> into half equally, and show it to him as a <u>whole</u>, and then <u>divide</u> it in half...

7. I. That's exactly what I would do!

O.K.

Let me try another one.

You obviously know all about things like that...

[I. realizes he needs to reach for more sophisticated questions.]

Let's get to something more interesting...

How would you explain "two-thirds" to somebody

[I. writes $\frac{2}{3}$

on the pad of paper that they are sharing.]

To somebody who didn't know about fr ns?

8. H. Ahh... Well... you take something... ... a <u>whole</u>... and then you would divide it into <u>thirds</u>...

9. I. Right

10. H. [continuing] ... instead of <u>halves</u>...

11. I. Right... And when you say "divide it into thirds", you obviously mean "take three equal pieces", right?...

12. H. Um [agrees]

13. I. And then how would you show them <u>two</u> thirds, as opposed to <u>one</u> third, or some other number of thirds? [silence]

[I. again tries to establish a more concrete task setting, by reviewing what they have discussed.]

Let's see: you took the whole, and divided it into three equal parts... and then what did you do?

14. H. Hmm... I don't know...

15. I. You may have said it, and I may have missed it...

16. [pause]

17. H. I don't really know what I did...

18. I. Well, let me try another one... How would you explain "two fifths"?

[I. writes $\frac{2}{5}$.]

19. H. O.K.... You take another whole <u>piece</u>, and you divide it into fifths...

20. I. Right

21. H. [continuing]... five equal pieces...

22. I. That's what I would do...

Now what would you do?

23. H. Ahmm... You take the other... the other two ... or three, or whatever we did, and show them the difference

24. [I. suspects that H. is trying somehow to combine or compare the "two thirds" and the "two fifths", and is getting herself confused. He tries to separate the two problems.]

25. I. Well... without worrying about those, just focus on the two fifths...

26. [brief silence]

27. I. We took a whole, right? [H. Right] I wish I had something here we could really divide. I'll try to bring something next time. [As he talks, he draws a rectangle on the paper, and divides it into fifths -- roughly like five white Cuisenaire rods in a row.]

Let's say we... one, two, three, four, five -- pretend those are equal [H. Um-hm [agrees]] -- I didn't quite get it right. So -- I've got five equal pieces. Now, what would you do to show him two fifths?

28. H. Probably shade in ... take away... two of the ... two of the ...

29. I. Show him those two. [He verbalizes H.'s gestures.] Perfect.

[30. Note: The transcript makes this sound ambiguous -- but H's gestures and inflections made it clear that she was showing someone the two fifths, NOT the residual three fifths.]

31. I. Okey-doke. And I know what you'd say for "three-fourths," I guess. [Decides to check:] What would you say for "three fourths"? [Writes $\frac{3}{4}$.]

32. H. Divide a whole into four equal pieces, and shade in, or take away, three of them.

33. I. [I. wishes to clarify the ambiguity between what you "take" and what you "leave" or "ignore".] You'd give him three of them? [H. Hmm (agreeing)]. So you'd be giving him three-fourths.

How would you show him or her five fourths?

[I. writes $\frac{5}{4}$.]

34. H. Well... You'd probably divide it into ...

[pause]

I don't know [she is surprised to discover that she cannot think of an answer].

[pause]

I'm just starting fractions and numbers and all that, so...

35. I. O.K.

What's another...

[Writes $\frac{1}{4}$.]

This is known as "one fourth".

36. H. Um-hm [agrees]

37. I. What's another name for that?

38. H. [confidently] A quarter

39. I. Yeah. A quarter.

40. I. How many quarters are there?

41. H. Four

42. I. Could there be more than four quarters?

43. H. Um-hmm [meaning "Yes,certainly."]

44. I. Yeah. Surely could. I was going to show you [he had a pocketful of quarters], if you'd said "No". [He takes ten quarters from his pocket, anyhow.] That coin is called "a quarter". Why is it

called "a quarter"?

45. H. Because it's a quarter of a dollar.

46. I. That's right, of course [I.'s tone of voice apologizes for asking such trivial questions; in fact, H. is clearly quite sophisticated and quite well able to hold her own in a discussion of this sort, and I. does not want her to feel that he is "talking down" to her.]

47. I. Why is it that I can so easily show you five quarters there? [He separates out 5 quarters.]

48. H. Ahmm... [pause]

I'm not so sure. [She is both surprised, and amused, to find that here is something she perhaps does not understand.]

[Longer pause]

49. I. Can you say what makes the five...

[Interrupts himself.]

You had no trouble at all with one-half or two-thirds or two-fifths or one-fourth. [H. Mmm (agrees; they were easy)] What makes five fourths harder than that?

50. H. Probably nothing...[She is, politely, annoyed with herself for not being able to give an instant answer. (Does this suggest that H's typical experiences in school have not included many thought-provoking questions? She certainly expects herself to know every answer immediately and effortlessly.)]

[pause]

Hmm...

[pause]

I'm kind of mixed up about this...

51. I. Can you _say_ what it is that you're "mixed up" about? You don't seem to me very mixed up about much of anything, actually!

52. H. ...It's just taking me a minute to get _arranged_ in my brain... I don't know... This _seems_ to be much more complicated.

53. I. Can you say _why_ it's more complicated?

54. H. [Laughs; she sounds genuinely amused, not embarrassed] No!

55. I. You know, that sometimes _helps_ a _lot_... If you can decide what's making it complicated, then you can probably fix it.

56. H. Probably that it's FIVE fourths, and not four, or any number less than four... [H's formulation of this answer is an accurate indicator of her very considerable sophistication]

57. I. Yeah. I think that's _exactly_ right!

And why would it be _easier_ if it were _three_ fourths?

58. H. Because it would be a...

[pause]

Well, this is an _improper_ fraction, right? [I. Right] So, it would be a proper fraction, I guess [if it were three fourths]. It's just that...[pause] I've done more with proper fractions... than I have with...

See, ... I never... In my other school I'd never even done fractions, previously... [And this is an alert, bright, sophisticated fifth grader in June, at the _end_ of the fifth grade year!]

So... I don't...

59. I. Unh-huh. [friendly acceptance]

Well... um... O.K.... Let's see...

I think that's an interesting question, because I don't ... There obviously _is_ something that makes five fourths a little bit harder. Everybody thinks it's harder.

But I don't know that it ought to be, really.

It's not *any* trouble... [Interrupts himself to try for a closer communication to representations in H.'s mind.] You agree that I don't have any trouble *showing* you "five quarters," in the sense of that money, there?

60. H. Hmm. [agrees; showing five quarters was not difficult]

61. I. How would you show somebody five quarters if you wanted to go back to something we've divided up [I. draws as he talks -- a rectangle divided into 4 parts, rather like 4 white Cuisenaire rods in a row.] If I use...

[I. interrupts himself, making sure:] You wouldn't have any trouble at all showing somebody five quarters with money, right?

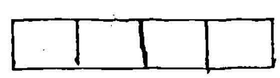
62. H. Um-hm [agrees; she could do it easily]

63. I. Suppose we tried to represent quarters...[still drawing as he talks] I can get quarters better than I can get fifths, because I can divide it [the rectangle] into half, and then I can divide the halves into half.

So... there!... I've got quarters. [displaying the paper]

How could you show me five quarters?

[Remark: The paper shows



Of course, at this point you CANNOT show 5 quarters. Something must be added!]

64. H. You could add another quarter...

[65. Remark: This is *half* of the crucial truth here. You MUST have *more*. The other half of the crucial truth is the realization that making 5 *pieces* won't help -- you will just have 5 fifths, instead of 4 fourths. In the process of "getting more", *you must not*

change the UNIT. But the interviewer didn't dare test to see if H. had this second part of the story figured out. It involves a meta-analysis that H. probably had never carried through.

So -- instead of probing for H.'s understanding of the role of the "unit", I. chose to state his questions in a way that he hoped would skirt the issue, and avoid confusion.]

66. I. O.K.

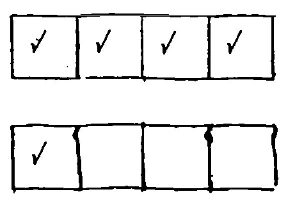
Indeed.

And, in :t, what people often do it to imagine another whole thing down here...

[draws a second rectangle, intended to be congruent to the previous one]

...and then divide that into fourths...

[I. now places check marks in 5 of the "quarters"



]

and say 'There's one, two, three, four, five'! That's really what it is!

67. H. Um-hm [She appears to accept this as a solution.]

68. I. Now... somebody could come along, and look at what we have here, and say "You've got one, two, three, four, five, six, seven, eight... So! Why aren't those eighths?"

That's sort of a good question.

Let's try it another way. [I. counts out 8 25-cent pieces ("quarters").] There are eight of these coins -- so why don't I call them "eighths"?

69. H. [Happily, H. sees the same pattern in both the rectangles and in the money!] Because, in these groups right here, there's only four!

70. I. Nifty! That's nifty! [So -- H. has at least the rudiments of the concept of "unit"!]

That's just right.

I want to repeat what you just said. You said "Because in this group right here there's only four." [gestures of H. and I. both indicate the first rectangle and the first dollar's worth of coins.]

What's the key thing?

What is it that we really divided up, here, to get those fourths, when we called those "quarters"?

What did we divide up?

71. H. A dollar

72. I. [inflection indicates agreement] a dollar ... and [gesturing to the next stack of 4 quarters] this was a different dollar that we divided up.

And so when we were dealing with these rectangles, we went on to another rectangle, a different rectangle...

That's exactly the answer.

That's really very nice.

[73. Remark: Now that the foundation of the concept seems secure, I. decides to attach the word "unit" to it.]

74. I. Do you know the word "unit"?

A word mathematicians sometimes use...

I'll write it.

[Writes: "unit"]

75. H. I've heard of it... but never really used it.

76. I. O.K.

People would sometimes say: "We use the dollar as the <u>unit</u> here."

I've forgotten the exact phrase you used a moment ago -- you had a very nice phrase for it. Something like: "the thing we divided up." Or maybe you said "This group right here." Whatever...

What you called "this group right here" is what mathematicians would call "the <u>unit</u>". So, if we take the dollar as the unit, then these coins <u>ought</u> to be called quarters.

77. I. Suppose, now, we wanted to talk about pizzas.

[Interrupts himself, once again making sure of closer contact with the student's ideas:]

Can you draw a pizza?

78. H. Without the topping?

79. I. [Laughs] Yes. I think so, yes!

80. H. [draws a circle]

81. I. O.K. And now, if you wanted to show people five fourths with that, what would you do?

What is it we're using as the unit?

82. H. The pizza.

83. I. Yeah. Whole pizzas.

So... if you wanted to show people five fourths, how many units would you want?

84. H. Two. [very quickly and confidently; she knows!]

85. I. Two. <u>Exactly</u> right!

I don't think you're very confused about that at all! I think you've been very clear about it.

Do you think of yourself as good at mathematics?

86. H. Yah! [confident and happy]

87. I. Yeah. I think you're very good at mathematics.

88. H. I used to be pretty horrible at it. I'm pretty good at it now. [Once again, perhaps, an allusion to this deficient school that H. used to attend -- wherever that was. The interviewer preferred not to ask.]

89. I. What made the difference?

90. H. I don't know.
[So... maybe H. was NOT alluding to "that other school"...]
[pause]

VIII. Mapping and Retrieval Errors

The postulated model of information processing that we are using suggests several likely errors, including:

i) cues might trigger the retrieval of an incorrectly-chosen representation form;

ii) input information might be incorrectly mapped into representation slots;

iii) subjects may fail to carry out the steps of checking the appropriateness of retrievals and mappings.

iv) subjects may fail to build up a correct representation for the problem setting, or for the solution knowledge. Both of these are construction tasks, often involving the retrieval of component parts which must be carefully assembled together so as to get an adequate problem representation and an adequate "knowledge" or "solution" representation. [Cf., e.g., Davis 1982]

In fact, all four error types occur frequently in our collection of interviews. The third -- failure to check for the appropriateness of retrievals and mappings -- is so common that one might say that elementary school students (at least those whom we've interviewed) in the absence of explicit doubts raised by the teacher, simply do not make such checks. [Cf., e.g., Davis and McKnight, 1980.] This is by no means a new result. [Cf., e.g., Erlwanger, 1973.]

The first processing error can be seen, for example, underlying the behavioral error of responding with "6 divided into halves" -- hence $6 \div 2 = 3$, or $6 \times \frac{1}{2} = 3$ -- when the task actually calls for "6 divided by one half" (i.e., $6 \div \frac{1}{2} = 12$). In our interviews, this error is more common than the correct answer is, both among adults and among school children.

Compare also the following interview excerpt, involving T., who is a boy about to complete the year's work in grade 5:

77. I. [getting out some Cuisenaire rods]: I want to give names to these rods, here.

Suppose I want to call the red rod "one"... What name would you give to some <u>other</u> rod, if I want to call the red rod "one." [I. prefers to give questions in the <u>least-structured form</u> possible -- hence he does NOT direct T. towards any specific rod.]

[78. T. picks up a light green rod]

79. I. [describing the action for the sake of the audio tape] O.K. You picked up a light green rod. What will we call that?

80. T. Two? [There is a question suggested by his inflection.]

We have used this question with several hundred students in grades two through five, and about 25% of the students give this same "wrong" answer: "if the red rod is called 'one,' then the light green rod should be called 'two '." But is the answer really wrong? There are at least two mathematical structures that every student has in mind, either of which could be retrieved as a representation form for this problem:

(i) size (or, more specifically, <u>lengths</u>)

(ii) sequential order.

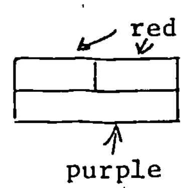
The positive integers have a sequential order -- 1, 2, 3, 4, ... -- just as the letters of the alphabet do -- A, B, C, D, ... -- and just as the Cuisenaire rods do -- white, red, light green, purple, yellow, ... If this "sequential order" structure is retrieved, and if the teacher begins the mapping by saying that red is to be called "one," then the first few steps in the mapping should go as follows:

red	⟷	one
light green	⟷	two
purple	⟷	three
yellow	⟷	four

and so on. This mapping does NOT contain any inherent internal contradiction, and is therefore "correct" -- at least in this simple sense.

If one performs more complex evaluation procedures, however, this mapping appears to be unsatisfactory. Specifically, although many sequential structures (such as the alphabet) have ONLY sequential order, some of them allow other possibilities. Numbers and lengths both allow the possibility of addition -- one can put two Cuisenaire rods end-to-end to make a train (as in some earlier pictures in this report). One can therefore ask: is this mapping also correct if we apply the criterion of "addition"?

Clearly it is not. "Red plus red" is a train whose length matches the single purple rod



,

but if we call red "one" and purple "three," this translates into

"1 + 1 = 3".

Left to his own devices, T. does not perform this check, and accepts the incorrect mapping:

red ⟷ 1
light green ⟷ 2
purple ⟷ 3 .

[We shall see later, however, that a slight scepticism from the teacher is enough to trigger some re-assessments by T.]

The correct structure to retrieve, of course, is the one that deals with size or length. Here, if red is one, the mapping necessarily becomes:

white ⟷ one half
red ⟷ one
light green ⟷ one and one half
purple ⟷ two

and so on. This mapping is an isomorphism when we consider the operation of addition:

"red + red = purple"

becomes

"one plus one equals two", and so on.

Concerning the very common phenomenon of a student not checking to see whether retrievals and mappings are appropriate, refer to the complete interview with T. On at least four occasions during the interview he makes errors which he was capable of discovering and correcting -- but on none of these occasions does he discover the error by himself. On the first occasion [utterances 77-81] we have seen him make the inappropriate choice of saying that, if the red rod is called "one," then the light green rod should be called "two." If we follow the interview just one item further, however, we see that T. is able to find and correct errors quickly -- provided the interviewer gives some suggestion that there MIGHT be a possible error. Without such a cue, T. apparently does not bother checking.

The phenomenon of "mapping errors" is not new. Rosnick and Clement [1980] report this in the case of a student ["Peter"] who maps "the number of English people" into C, and "the number of Chinese" into E, reversing his earlier convention (which was represented by a very clear mnemonic, namely, the initial letter of each key word!), in order to preserve an incorrect mental representation structure that he was using. [Cf. Rosnick and Clement, 1980, pp. 11-12; cf. also Davis, 1980.]

A particularly interesting example of the process of using an incorrect mapping of input data into frame slots occurred in an interview of E_c, an eighty-seven year old retired woman who had worked for 55 years as a legal secretary. Indeed, we see not merely a wrong mapping, but the more interesting phenomenon of a correct mapping that was changed to an incorrect mapping.

1. I. Let me write it...

[writes:

$$\frac{1}{2}$$]

Suppose you were trying to explain to somebody what "one half" meant -- how would you explain it?

2. E. You take a.... a whole, and ... divide it in ... in two parts.

3. I. O.K. How would you explain what "two thirds" meant?

[writes

$$\frac{2}{3}$$]

4. E. I would take something and make ... make three piles... and take... two of them...

5. I. O.K. That's just what I would do.

How would you explain something like "thirteen fifths" to somebody, if they didn't know about it?

[writes: $\frac{13}{5}$.]

6. E. ...Well...I...That's...that's a sticker!

7. I. Yeah, it is.

[8. pause -- silence on the tape]

[9. Comment: Given our postulated model, we interpret this data as a likely attempt by E. to retrieve the same representation structure (or "frame") that she used in utterances 2 and 4 -- which explains

$$\frac{a}{b}$$

by taking something, dividing it into b parts, then taking a of these parts. She then tries to map "5" into "total number of parts", and "13" into "number of parts to be taken". But of course this mapping fails so self-evidently that the failure cannot be ignored. This failed mapping of input data into slots in the knowledge representation

structure leads E. into the confusion that she reveals in utterance No. 6.]

10. E. That's supposed to be thirteen divided by five, or just "thirteen fifths"?

[11. Comment: E. is the first subject interviewed to bring up, on her own, this important distinction. We interpret this as indicating that E. does more evaluating of the correctness of a representation, and more comparisons among possible alternative representations. Life experience may explain part of this. E. is in her eighties, and for fifty-five years she worked as a legal secretary and tax specialist. Now retired, she was the first adult to be interviewed in our study.]

12. I. Well... it comes out to be the same thing, actually. [This was probably an inadvisable remark for I. to make; conceptually the two are very different, indeed -- or may be, depending upon the definitions you use.]

[13. pause -- silence on tape]

14. E. I don't know how I would do that.

[15. brief pause]

16. E. I would make thirteen piles,...

...and take five of them

...and see how many I had left.

But that isn't quite right, either. I don't know.

[17. Comment: At least four interesting things have happened here:

(i) First, E. reverses the mapping of input data into frame slots, so that, instead of mapping

"5" ⟶ "total no. of parts"

"13" ⟶ "number of these parts that you take"

--which, of course, fails [but is fundamentally the correct pattern, provided one introduces the concept of unit] -- she uses the mapping

"13" → "total no. of parts"

"5" → "the number of these parts that you take"

(ii) For some reason -- presumably as the result of an internal evaluation -- she rejects the step of completing the a/b representation, which would now call for $\frac{5}{13}$.

Presumably something about

$$\frac{5}{13}$$

alerts E. to the existence of some sort of difficulty, contradiction, or error.

(iii) E. now switches to a different knowledge representation structure (or "frame"). Specifically, she switches to the representation structure for subtraction ("...and see how many I had left.")

(iv) True to her typical behavior, E. -- unlike our elementary school students -- carries out another internal evaulation, and says, "But that isn't quite right, either."

For E., this is what one learns to expect. For most of our subjects, this would be highly unusual.]

18. I. Well... O.K....

19. E. [interrupting] How would you divide that, really? [She is genuinely curious.]

20. I. Well... You'd do just the same thing you did up here [gestures to where he has written "$\frac{1}{2}$" and "$\frac{2}{3}$" on the paper].
You'd take something -- whatever -- a pie, or whatever it is [he draws a circle] -- and divide it into five pieces -- it would be like that -- now, I'll need several pies [draws two more circles (or "pies"), for a total of three]... ...and I divide each of them into five equal pieces [draws this, more or less] -- let's see, I've got ten pieces there, so I need the third pie [draws dividing lines in the "third pie"]... now, I can take 5, 10, 11, 12, [coloring them in], thirteen

pieces, and each piece is one fifth -- one fifth of a pizza, or pie, or whatever this is -- O.K. Is that all right?

21. E. Oh. [inflection suggest partial comprehension beginning to dawn]

22. I. ...and that's really what you did up here [gestures to "$\frac{1}{2}$" and "$\frac{2}{3}$", as previously written on the paper]. You took a <u>unit</u>, and divided it up. You took a unit, and divided it up into three parts.

23. E. Mmm. [inflection suggests growing degree of comprehension]

24. I. I try to get students to look <u>first</u> at the <u>denominator</u>, and you did that up here. You took something and divided it into two pieces [for $\frac{1}{2}$]. You took something and divided it into <u>three</u> pieces [for $\frac{1}{3}$]. So, for

$$\frac{13}{5},$$

you take something and divide it up into 5 pieces. Well, that's not going to be enough to give you 13 pieces. So, then -- if this is, say, pizza -- you need to have <u>another</u> pizza. That still won't do it. That'll give you 10 pieces. Each of those parts is a <u>fifth</u>, all right, because you're dividing each pizza into 5 equal pieces -- a fifth of a pizza -- but that only gives you ten, so you need to get a <u>third</u> pizza, and then you can take thirteen. So that does it!

25. E. Um-hm. [Reasonably satisfied]

[26. Remark: E. shows a pattern which most people, including experts, probably use often. If one mapping of input data into frame slots fails, switch some things around to get a new mapping, and see if that will work. It resembles the way one might insert plugs into sockets in connecting up electrical equipment -- or the way one might reverse a key if, on the first try, it won't fit into a keyhole. This is probably a very valuable procedure.

Unfortunately, it can lead us astray. In some cases, as here, it can lead to a violation of the meaning of the frame. After all,

$$\frac{13}{5}$$

is NOT the same thing as

$$\frac{5}{13} .$$

[Note that the switch of "5 + 2" in place of "2 + 5", reported by Resnick [1983], is another instance of this same device, but in this case it succeeds, thanks to the commutative law for addition.]

It is our opinion that schools do far too little with the important idea of Extension of Systems [cf. Davis, Explorations in Mathematics, A Text for Teachers, Addison-Wesley, 1967.] -- that is to say, the process of analyzing a concept so as to determine how best to extend it beyond its present domain. One must do this to extend

$$2^2 = 2 \times 2 = 4$$

$$2^3 = 2 \times 2 \times 2 = 8$$

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

into

$$2^{\text{[illegible]}} =$$

$$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

and even if

$$2^{1/2} = \sqrt{2}$$

$$2^{1/3} \quad \sqrt[3]{2}$$

or even

$$e^{\pi i} + 1 = 0.$$

One does it in Abelian and Cesaro summability for infinite series, for Poincare's asymptotic series, for the extension of "multiplication as successive addition"

$$3 \times 2 = 2 + 2 + 2 = 6$$

into

$$3 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5$$

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{3 \times 2} = \frac{1}{6}$$

and even into

$$(1 + i)(1 - i) = 1 - i^2 = 1 - (-1) = 2 .$$

One does it with *factorials*:

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 3 \times 1 = 24$$

or

$$n! = n \, [\, (n - 1)! \,]$$

and the important extension to

$$0! = 1,$$

with subsequent extensions, via Euler's work, into

$$\frac{1}{2}! = \frac{1}{2} \pi$$

In the present case, we do this when we extend the idea of *proper fraction* into the idea of *improper fraction* by taking several separate pizza pies, any *one* of which is now conceived of as a *unit*. This preserves the essential meaning of fraction.

In short, what E. needed to do -- instead of switching her *mapping* of input data into frame slots -- was to analyze the concept to identify its essence, and then to carry out a correct *extension* of that concept, so as to enlarge its domain.

Schools hardly deal at all with this important matter.

As a final instance of an incorrect mapping, we use an interview from outside the area of fractions. This interview is interesting for two reasons: first, it shows how the effect of "cycling through" our postulated model many times can manifest itself. Because of this repeated cycling through the same steps, what is, on an earlier cycle-through a mapping of input data into the slots of a knowledge representation becomes, subsequently, an evaluation of success, a rejection, a meta-analysis, a revised mapping, and, subsequently, a step in building a more appropriate representation structure.

This interview is also interesting in that the subject had solved the problem before the actual interview, which therefore could deal only with after-the-fact reporting of what had taken place. This is by no means our first-choice method, given a choice, but of course in many situations one does not have the freedom to choose. With more difficult problems it is not uncommon for a subject to think about them for hours, or even for days, and a cognitive breakthrough may occur at any time.

The subject was a junior-year university student majoring in computer science, with a fairly strong background in calculus, but essentially no prior knowledge of abstract algebra. He had studied Section 2.2 of Dornhoff and Hohn, Applied Modern Algebra, but at the time of the interview had not yet read Section 2.4. He knew that a relation from a finite set

$$A = \{A_1, A_2, \ldots, A_n\}$$

to a finite set

$$B = \{B_1, B_2, \ldots, B_m\}$$

is a set of ordered pairs

$$\mathcal{R} = \{(A_i, B_k), (A_p, B_q), \ldots (A_r, B_s)\}$$

where all the A_x are elements of set A, and all the B_y are elements of set B.

Section 2.2 of Dornhoff and Hohn shows how to represent any relation as an mxn matrix M, with entries m_{ij}, where the entry $m_{ij} = 1$ if $(A_i, B_j) \in \mathcal{R}$, and $m_{ij} = 0$ otherwise. At this point, he decided to attempt to solve Problem 4, on page 51 of Dornhoff and Hohn. This problem presents the student with a relation

$\mathcal{R}$, represented by its matrix M (as above), and asks the student to determine whether the relation $\mathcal{R}$ (which maps a set of four elements into itself) is transitive. Transitivity is defined as:

> The relation $\mathcal{R}$ is transitive if, whenever $(X,Y) \in \mathcal{R}$, and $(Y,Z) \in \mathcal{R}$, then $(X,Z) \in \mathcal{R}$.

This is all very abstract, and the subject, C., believed that he must somehow construct a more congenial mental representation for the relevant knowledge that would allow him to think more effectively about this task. During the first interview he was not able to do so, and reported that, having no suitable representation, he could not think about the problem. He said he would have to go away, do other things, and see what might occur to him. During this first session he did, however, conjecture and prove that the composite relation $\mathcal{T} \equiv \mathcal{R} \circ \mathcal{S}$ had a matrix representation that could be obtained from the matrices of $\mathcal{R}$ and $\mathcal{S}$ by ordinary matrix multiplication, subject to the rule that $1 + 1 = 1$ (which makes sense when set inclusion is the subject of interest, because once an element is included in a set, it's in, and that's all there is to it -- set inclusion might be said to work like citizenship in the U.S.A.; if you're a citizen, you are, and additional claims to U.S. citizenship do not effect your status).

Part of the value of this example lies in the fact that many readers may be able to place themselves in a role similar to the student's, something that we cannot usually do when we speak of fractions. Our own representations for fraction knowledge are usually so good that we are hardly aware when we use them. In a situation where we do not have adequate representations everything looks quite different -- weird, or meaningless, in many cases. The reader to whom this present task seems hopelessly abstract is fortunate -- he or she can see more clearly the need to create a mental representation quite different from the one that is presented in the book.

During the first session, C. used the valuable heuristic strategy of trying out some simple, explicit cases. One case that he made up, and [illegible] as this:

he defined the set A as $A = \{2,3,4,6\}$,

and the relation $\mathcal{R}$ as divisibility, written

$$a \mid b.$$

Now, clearly, divisibility is already known to be transitive:

$a|b$ and $b|c$ implies $a|c$. In the set A, we have $2|4$, $2|6$, $3|6$, $2|2$, $3|3$, $4|4$, and $6|6$. This gives us the M matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

C. computed M x M:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which seemed to him to be relevant, but he wasn't sure how. Because the transitive requirement has two "if" statements:

If aRb and if bRc, then aRc,

C. felt that only the presence of 1's in certain matrix locations could create any requirements. If one or both of the key entries were 0, the "if" clauses were not satisfied, and no requirement had to be met.

The trouble was, C. could not figure out where, in the matrix, the key 0's and 1's might be hiding.

To make matters worse, because of the nature of matrix multiplication, the key 1's might be in any of several possible locations. This accurately reflects the requirement that, in order to have transitivity, (a,c) must belong to $\mathcal{R}$ if there exists an element b such that (a,b) and (b,c) both belong to $\mathcal{R}$. [Thus, the element b is a kind of "missing link" -- or "elusive link." You are "a friend of a friend of," say, Pete Rose, if there exists a person whom you know, who knows Pete Rose. But are you sure that none of your friends knows Pete Rose? That might require quite a bit of checking.]

C. left the first interview, saying that he needed to think further about this problem.

In the second interview, on the following day, C. reported that he had solved the problem. $\mathcal{R}$ is transitive if and only if, whenever M x M has a 1 in position m_{ij}, then M does also.

What had previously confused C. had been a mapping error. Whenever he computed $M \times M = M^2$, he had at first believed that a 1 in the m_{ij} position in M^2 corresponded to the relation aRc; that is, he had unconsciously made the naive sequential map:

1 a 1 here, say

$$(M) \quad (M) \quad = \quad (M^2)$$

would correspond to the right hand element here: $\updownarrow$

aRb bRc aRc

But this, of course, is incorrect. The presence of a 1 at a position in M^2 testifies to the existence of the missing link 1's in appropriate locations in the two factors M on the left, and hence to the first two terms in the transitivity statement:

$$\left(\xrightarrow{\text{same row}} (M)\right) \quad \left((M)\downarrow \text{same column}\right) \quad = \quad (M^2 \quad 1)$$

a 1 here

corresponds to the existence of "b"

aRb and bRc

and therefore alerts us to the need for aRc, which means that the original matrix M must have a "1" in this location, in order for $\mathcal{R}$ to be transitive.

That is to say, a correct correspondence looks like this

$$\begin{pmatrix} (M) \end{pmatrix} \quad \begin{pmatrix} (M) \end{pmatrix} = \begin{pmatrix} & 1 \\ M^2 & \end{pmatrix} \text{ a "1" here}$$

corresponds to the existence of b

aRb and bRc must imply aRc ,

His earlier, erroneous, map had looked like this:

$$\begin{pmatrix} (M) \end{pmatrix} \quad \begin{pmatrix} (M) \end{pmatrix} = \begin{pmatrix} & 1 \\ M^2 & \end{pmatrix} \text{ a "1" here}$$

aRb and bRc must imply aRc .

Notice that, by the time that C. had figured this out, he had cycled through our postulated sequence

cueing

retrieval or construction of a representation

mapping input data into slots in the representation

judging the correctness of representation and mapping

many times, with the result that solving the mapping problem had, in fact, actually contributed to the construction of the final representation.

[During this process, C. had considered all possible 2 x 2 matrices, and determined that it was not necessary for M^2 to be identical with M; his earlier example of the divisibility relation happened to have this special property, but it was not a general requirement. Thus, C.'s use of the heuristic "try easy, specific cases" had ultimately paid off.]

We had a third interview with C., on the following day. He was not clear on what he had done, and had to refer to some notes he had written the day before (from which we have taken the "correct" diagram above). This interests us, because C. was very impressed with what he had done, excited by it, and proud of it. He had not treated it casually. Yet, in order to reconstruct it one

day later, he nonetheless had to refer to written notes. [With the aid of the notes, his reconstruction was correct.] We have long suspected that the construction of elaborate, complex representation structures, within one's own mind is a very demanding task, by no means easily accomplished.

Some expert chess players deliberately teach themselves to play "mental chess" -- that is to say, to play a game where no physical board is used. Both players must keep the board position in mind, without help from any visual or tactile inputs. One expert, Boris Siff, described for us how he set about learning to play mental chess. First, he set himself the task of learning to visualize a board that was merely 2-by-2 (as opposed to the 8-by-8 board of actual chess). He would visualize each of the chess pieces on this 2-by-2 board, visualize possible moves, possible captures, etc. When he was confident that he could visualize all possible play on the 2-by-2 board, he extended his mental board to 3-by-3, and repeated the process. He continued this "extending" process until he could deal effectively with the full 8-by-8 board. This strikes us as one more piece of evidence concerning the difficulty of creating adequate mental representaions for complex situations.

Do typical educational programs recognize the importance and difficulty of this task? Building good mental representations seems to be <u>hard</u>. Few school programs appear to recognize this difficulty, and to provide for it. If, as growing evidence suggests, these mental representations are synthesized partly by combining <u>actual tactile and visual experience</u>, one would expect school mathematics programs to abound in manipulateable materials such as geoboards, Dienes' MAB blocks, Cuisenaire rods, trundle wheels, or even pebbles, bottle caps, graph paper, drawing materials, and so on. This hardly seems to be the case. After more than two decades of demonstrations of the values of such materials, they are hardly used at all, and even where they are present they are not usually employed with understanding. Schools, instead, treat mathematics as what Paul Johnson (of UCLA) has called "the dance of the digits." That is to say, they

regard mathematics as something written on paper. In so doing, they confuse the notations for recording mathematical results with the quite different -- and more important -- processes of thinking about mathematics, an activity that takes place within the human mind and is rarely recorded on paper.

IX. Results

Four categories of results come from this work: (i) results that deal with human information processing; (ii) results that deal with mathematical topics; (iii) inferences about school programs; (iv) an assessment of the research methodology that has been employed.

It is worth pointing out that interview studies yield a great wealth of information that does not lend itself to easy summarization. In a sense, the "result" is the corpus of interviews themselves, with specific analyses, just as the "result" of Beethoven's composing is the body of his works (plus, perhaps, its impact on subsequent composers). With this caution, we can proceed to identify some prominent patterns:

A. Results that deal with Information Processing

1. Nearly all students have access to good processing capabilities when dealing with concrete materials.

    a) To make this more definite, we list some of the specific kinds of processing that are involved:

        i) Recognize a length as "too long," "too short," or "just right." [as in the case of fitting together "trains" of Cuisenaire rods];

        ii) Estimate the size of quantities [cf. Interview no. 1 ("H"), utterance 216];

        iii) "Add" by putting lengths end-to-end [cf. Interview no. 1 ("H"), utterances 221-3];

iv) Divide an interval into n equal parts, for $n \leq 10$ [cf. Interview no. 1 ("H"), utterances 229-240];

v) Select a length (with Cuisenaire rods) to use as a unit, such that one can find halves, thirds, or fourths [cf. Interview no. 1, ("H"), utterances 191-203, and 209-211];

vi) Determine quantities such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and so on [cf. Interview no. 1 ("H"), utterances 198-209];

vii) Modify your language to go from one (usually tacit) "unit" to the extension of using several units, in order to deal with improper fractions such as $\frac{7}{4}$;

viii) Devise sub-goals, sequenced so as to achieve some larger goal;

ix) Describe, in words, what you have just done, and why you did it. [These actions of students are highly suggestive of Winston's "blocks world" (cf. Minsky and Papert, 1972)]

x) Look at two actions, or two situations, and describe how they are different.

2. [This is really an interpretation of results.] Consequently, a major task of school programs is to build on these processing capabilities (as in 1, above), and use them as a foundation for symbolic mathematics that involves notations written on paper.

3. [Another interpretation.] The strength in processing described in 1, above, is due in part to the fact that students have good mental representations for such things as "the length of a Cuisenaire rod", and for such processes as "putting two rods together end-to-end."

4. Most students have at least moderately good ability to transfer this concrete processing capability (described in 1, above) to somewhat more abstract contexts. [Most of our interviews demonstrate this process at work, because most interviews were set up to make this possible.]

Note: A *mental representation* is sometimes called a *frame*, after Minsky, 1975. Recall that a representation or frame will usually have some "slots" into which input data must be mapped. Thus, if some cue in a problem statement triggered the retrieval of, say, a "subtraction" frame, that frame, in effect, *asks* for certain data, which we might suggest by imagining a form, with blanks to be filled in:

What is the minuend?	____________
What is the subtrahend?	____________

This illustration is of course, only a metaphor. We do not mean to suggest that words such as "minuend" and "subtrahend" would be employed. Indeed, most students do not know this word. But, with greater or lesser clarity, they *do* know these *concepts*.

5. Both students and adults are skillful at rejecting one mapping of input data into frame slots, and replacing that mapping by some new mapping. Younger students, however, are *not* very good at detecting when one mapping is unsatisfactory, and needs to be replaced by another.

6. Apparently younger students' failures to detect incorrect mappings are at least partly due to the small amount of *checking* or "evaluation of mappings" that younger students do.

7. The adults in our sample were much more likely to carry out such checking, and hence more likely to reject an incorrect mapping. (But this does not mean that they regularly end up with correct mappings. Frequently the new mapping will also contain errors. Cf., for example, the interview dealing with the improper fraction

$$\frac{13}{5} ,$$

with the retired secretary, E.)

8. When a task requires a <u>meta-analysis</u> and an <u>extension of a system</u>, neither students nor adults in our sample do well. This is somewhat surprising, given that many adults can cope with "extension-of-system" problems very well in other contexts, as, for instance, in answering the question:

Who is the "First Lady" in Britain [in 1982]?

If the "First Lady" is initially defined as <u>the wife of the President of the United States</u>, then one must extend "United States" to mean "nation," one must extend "wife" to mean "spouse" (as one will have to do when we have a female president), and one must extend "President" to mean "head of the operative political government" (since presumably it is Margaret Thatcher, and not Queen Elizabeth, who is the British equivalent of the President of the United States). <u>Some</u> students are indeed very good at this in mathematical contexts -- one fifth-grade student added

$$\frac{1}{2} + \frac{1}{3}$$

by writing

$$\frac{1}{2} + \frac{1}{3} = \frac{1.5}{3} + \frac{1}{3} = \frac{2.5}{3} = \frac{\frac{5}{2}}{3} = \frac{5}{6}$$

-- but this is quite rare.

9. A theme that runs throughout this study, and two previous studies, is that <u>good mental representations are</u>:

a) <u>very important</u>

b) <u>very difficult for students to construct.</u>

[Cf., e.g., Young, 1982; Davis, Young, and McLoughlin, 1982.] As in the interview where T. continues to use "red is one," even after he had meant to change this crucial part of his representation, students often build representations that "weld together" some correct features with others that are not correct.

Neither the importance of mental representations, nor the difficulty in creating them, seem to be generally recognized; instead, school programs focus

undue attention on words, which are not at all the same thing.[1] You and I may both see the word "dog," but our mental representations of "dog" are almost certainly quite different, and when we think about dogs it is the mental representation that we use.

10. Among the students whom we observed, there was great variation in the extent to which problem-solving heuristics and sophisticated strategies were employed in dealing with symbolically-presented tasks. On the sophisticated extreme, we saw (above) the case of the fifth-grade student who used $\frac{1}{2} = \frac{1.5}{3}$ in order to add $\frac{1}{2} + \frac{1}{3}$. Similarly, a sixth-grade boy, asked to divide $\frac{5}{2} \div \frac{3}{8}$, made use of an integer algorithm that he knew, namely

$$\begin{array}{r|r l}
15 & \overline{1983} & \\
 & \underline{1500} & 100 \\
 & 483 & \\
 & \underline{450} & 30 \\
 & 33 & \\
 & \underline{30} & 2 \\
 & 3 &
\end{array}$$

so $1983 \div 15 = 132\frac{1}{5}$,

which he extended so that he could use it with fractions:

$$\begin{array}{r|r l}
\frac{3}{8} & \overline{\frac{5}{2}} & \\
 & \underline{\frac{3}{8}} & 1 \\
 & \frac{17}{8} & \\
 & \underline{\frac{15}{8}} & 5 \\
 & \frac{2}{8} &
\end{array}$$

whence $\frac{5}{2} \div \frac{3}{8} = 6\,\dfrac{\frac{2}{8}}{\frac{3}{8}} = 6\,\frac{2}{3}$.

[1] On the importance of representations, cf. this remark of Niels Bohr, one of the great physicists of the twentieth century: "When it comes to atoms, language can be used only as in poetry. The poet, too, is not nearly so concerned with describing facts as with creating images." Jacob Bronowski, The Ascent of Man, Boston, Massachusetts: Little, Brown & Co., 1973. p. 340

At the opposite extreme, many students seemed only to follow explicit instructions, and never to adjust adapt, invent, or plan.

B. Results that Deal with Topics

One topic stood out -- the idea of a unit. The ideal developmental sequence (in our view) would begin by dividing concrete entities into n equal parts and taking m of them, with $m \leq n$. At this stage, the role of unit is implicit and is usually left tacit. In subsequent lessons, this idea is extended so that one divides up, say, several pizza pies. At this point the idea of unit becomes central and must be made explicit. This becomes an important opportunity for meta-analysis: one can discuss why it becomes important to identify a unit. Apparently school programs do not follow such a path, or at least have very little success with it, for no elementary school student in our sample showed evidence of understanding why a unit needs to be identified, and how this relates to the interpretation of improper fractions. Nor -- and the interview with E., the retired secretary, is typical -- do adults do very much better.

C. Inferences About School Curricula

This study did NOT include planned observations of classroom lessons, so it has produced no direct data on what is taught, or on how it is taught. Nonetheless, striking similarities in the performance of students invite certain inferences about school curricula. With the preceding caveat, we list some of the most important of these:

1. School curricula do NOT make much use of the device of beginning with concrete tasks, then using these concrete tasks as the foundation from which abstract notations can arise.

To illustrate what we mean, place-value numerals can (and, in our view, should) arise as a "shorthand" for recording counting and combining experiences. At the earliest level, this work can be entirely concrete -- using, say, Dienes' MAB blocks, base 10. At the next level, tasks or games can be devised that require the recording of transactions; this

can be done pictorially, with a "Thou shall not have ten" rule that leads to exchanging 10 units for one long, 10 longs for one flat, etc. [Cf. Davis, 1983, for details.] Gradually, pictorial recording can be abbreviated, by a sequence of small modifications, until it becomes the standard notation of place-value numerals.

In a sense, this kind of development recapitulates human history -- one starts with a sensible task, and devises notations to help deal with it. These notations can then evolve toward greater sophistication and effectiveness.

That the schools in our sample do not use this approach comes as a surprise, because they DO have physical materials such as Cuisenaire rods and Dienes' MAB blocks. Apparently these are used mainly to <u>illustrate</u> mathematical techniques, and not as a source of <u>foundation</u> activities <u>from which abstract ideas and notations can develop gradually</u>.

2. It comes as no surprise, however, that school curricula fail to develop meta-analysis, the analysis of what you are doing and of what you have just done. A few rare students develop meta-analysis skills, mainly on their own, but this is not at all common. [Cf. Davis, Jockusch, and McKnight, 1978.]

3. School curricula seem not to develop the important "extension of systems" idea. [Cf. Davis, <u>Explorations in Mathematics. A Text for Teachers</u>, 1966.]

4. School curricula seem not to develop skill in <u>heuristic problem analysis</u>.

5. School curricula seem not to develop the habit of checking each key step or key result by comparison with other knowledge. Thus, a student <u>should</u> reject the addition

$$2 + \frac{1}{2} = \frac{2}{1} + \frac{1}{2} = \frac{3}{3} = 1$$

immediately on the grounds that $1 < 2$, and that the addition of positive

quantities does not behave like this.

6. In general, school curricula are NOT careful to help each student to develop effective mental representations for thinking about mathematics.

D. Inferences About This Research Technique

1. Study of the interview tape recordings convinces us that, by using such methods, one can very often get a good idea of how a student is thinking about some mathematical problem or topic. When more time is devoted to the increasingly careful analysis of such tapes, even more can be inferred.

2. The use of a postulated model of human information processing shifts the focus of many analyses in ways that we believe are beneficial.

3. It came as an unanticipated discovery when analysis of tapes showed clearly that the interviewers often interrupt themselves in order to move toward a better match between their mental representations and those of the students. We have since looked for this elsewhere, and found it to be a characteristic of effective teaching.

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March, 1983

The Development of the Concept of "Fraction" from Grade Two Through Grade Twelve

Robert B. Davis

FINAL REPORT OF

NIE G-80-0098

APPENDIX

Curriculum Laboratory, University of Illinois, Urbana/Champaign

1210 West Springfield Ave., Urbana, Illinois 61801

APPENDIX

FINAL REPORT OF

NIE G-80-0098

Friday, June 4, 1982

Kenwood School -- 10:00 a.m.

Heidi Rettig

Excellent episode -- at 1st (abstractly) she can't handle $\frac{1}{3} + \frac{1}{2}$ -- but later, with Cuisenaire rods, she gets it confidently.

1. I. O.K....We're thinking about fractions . . .

2. H. Um-hmm [agreement]

3. I. Let's take a very easy one -- let's take one half [writes $\frac{1}{2}$].

Suppose you wanted to explain "one-half" to somebody -- [interrupts himself, in order to establish a more concrete task setting:] Do you have any younger brothers or sisters?

4. H. I have a step-brother who's younger -- he's nine... eight or nine...

5. I. ...so he probably already knows about "one half"... but you can imagine a fairly young child who really didn't know about fractions?...

How would you explain "one half"? To him? Or to her?

6. H. Ah...Probably take some...something that can be divided into half equally, and show it to him as a whole, and then divide it in half...

7. I. That's exactly what I would do!

O.K.

Let me try another one.

You obviously know all about things like that...

[I. realizes he needs to reach for more sophisticated questions.]

Let's get to something more interesting...

How would you explain "two-thirds" to somebody?

[I. writes

$$\frac{2}{3}$$

on the pad of paper that they are sharing.]

To somebody who didn't know about fractions?

8. H. Ahh... Well... you take something... ... a <u>whole</u>... and then you would divide it into <u>thirds</u>...

9. I. Right

10. H. [continuing] ... instead of <u>halves</u>...

11. I. Right... And when you say "divide it into thirds", you obviously mean "take three equal pieces", right?...

12. H. Um [agrees]

13. I. And then how would you show them <u>two</u> thirds, as opposed to <u>one</u> third, or some other number of thirds? [silence]

[I. again tries to establish a more concrete task setting, by reviewing what they have discussed.]

Let's see: you took the whole, and divided it into three equal parts... and then what did you do?

14. H. Hmm... I don't know...

15. I. You may have said it, and I may have missed it...

16. [pause]

17. H. I don't really know what I did...

18. I. Well, let me try another one... How would you explain "two fifths"?

[I. writes $\frac{2}{5}$.]

19. H. O.K.... You take another whole <u>piece</u>, and you divide it into fifths...

20. I. Right

21. H. [continuing]... five equal pieces...

22. I. That's what I would do...

Now what would you do?

23. H. Ahmm... You take the other... the other two ... or three, or whatever we did, and show them the difference

24. [I. suspects that H. is trying somehow to combine or compare the "two thirds" and the "two fifths", and is getting herself confused. He tries to separate the two problems.]

25. I. Well... without worrying about those, just focus on the two fifths...

26. [brief silence]

27. I. We took a whole, right? [H. Right] I wish I had something here we could really divide. I'll try to bring something next time. [As he talks, he draws a rectangle on the paper, and divides it into fifths -- roughly like five white Cuisenaire rods in a row.]

Let's say we... one, two, three, four, five -- pretend those are equal [H. Um-hm [agrees]] -- I didn't quite get it right. So -- I've got five equal pieces. Now, what would you do to show him two fifths?

28. H. Probably shade in ... take away... two of the ... two of the ...

29. I. Show him those two. [He verbalizes H.'s gestures.] Perfect.

[30. Note: The transcript makes this sound ambiguous -- but H's gestures and inflections made it clear that she was showing someone the two fifths, NOT the residual three fifths.]

31. I. Okey-doke. And I know what you'd say for "three-fourths," I guess. [Decides to check:] What would you say for "three fourths"? [Writes $\frac{3}{4}$.].

32. H. Divide a whole into four equal pieces, and shade in, or take away, three of them.

33. I. [I. wishes to clarify the ambiguity between what you "take" and what you "leave" or "ignore".] You'd give him three of them? [H. Hmm (agreeing)]. So you'd be giving him three-fourths.

How would you show him or her five fourths?

[I. writes $\frac{5}{4}$.]

34. H. Well... You'd probably divide it into ...

[pause]

I don't know [she is surprised to discover that she cannot think of an answer].

[pause]

I'm just starting fractions and numbers and all that, so...

35. I. O.K.

What's another...

[Writes $\frac{1}{4}$.]

This is known as "one fourth".

36. H. Um-hm [agrees]

37. I. What's another name for that?

38. H. [confidently] A quarter

39. I. Yeah. A quarter.

40. I. How many quarters are there?

41. H. Four

42. I. Could there be more than four quarters?

43. H. Um-hmm [meaning "Yes,certainly."]

44. I. Yeah. Surely could. I was going to show you [he had a pocketful of quarters], if you'd said "No". [He takes ten quarters from his pocket, anyhow.] That coin is called "a quarter". Why is it

called "a quarter"?

45. H. Because it's a quarter of a dollar.

46. I. That's right, of course [I.'s tone of voice apologizes for asking such trivial questions; in fact, H. is clearly quite sophisticated and quite well able to hold her own in a discussion of this sort, and I. does not want her to feel that he is "talking down" to her.]

47. I. Why is it that I can so easily show you five quarters there? [He separates out 5 quarters.]

48. H. Ahmm... [pause]

I'm not so sure. [She is both surprised, and amused, to find that here is something she perhaps does not understand.]

[Longer pause]

49. I. Can you say what makes the five...

[Interrupts himself.]

You had no trouble at all with one-half or two-thirds or two-fifths or one-fourth. [H. Mmm (agrees; they were easy)] What makes five fourths harder than that?

50. H. Probably nothing...[She is, politely, annoyed <u>with herself</u> for not being able to give an instant answer. (Does this suggest that H's typical experiences in school have not included many thought-provoking questions? She certainly expects herself to know every answer immediately and effortlessly.)]

[pause]

Hmm...

[pause]

I'm kind of mixed up about this...

51. I. Can you say what it is that you're "mixed up" about? You don't seem to me very mixed up about much of anything, actually!

52. H. ...It's just taking me a minute to get arranged in my brain... I don't know... This seems to be much more complicated.

53. I. Can you say why it's more complicated?

54. H. [Laughs; she sounds genuinely amused, not embarrassed] No!

55. I. You know, that sometimes helps a lot... If you can decide what's making it complicated, then you can probably fix it.

56. H. Probably that it's FIVE fourths, and not four, or any number less than four... [H's formulation of this answer is an accurate indicator of her very considerable sophistication]

57. I. Yeah. I think that's exactly right!

And why would it be easier if it were th. fourths?

58. H. Because it would be a...

[pause]

Well, this is an improper fraction, right? [I. Right] So, it would be a proper fraction, I guess [if it were three fourths]. It's just that...[pause] I've done more with proper fractions... than I have with...

See, ... I never... In my other school I'd never even done fractions, previously... [And this is an alert, bright, sophisticated fifth grader in June, at the end of the fifth grade year!]

So... I don't...

59. I. Unh-huh. [Friendly acceptance]

Well... um... O.K.... Let's see...

I think that's an interesting question, because I don't ... There obviously is something that makes five fourths a little bit harder. Everybody thinks it's harder.

But I don't know that it ought to be, really.

It's not any trouble... [Interrupts himself to try for a closer communication to representations in H.'s mind.] You agree that I don't have any trouble showing you "five quarters," in the sense of that money, there?

60. H. Hmm. [agrees; showing five quarters was not difficult]

61. I. How would you show somebody five quarters if you wanted to go back to something we've divided up [I. draws as he talks -- a rectangle divided into 4 parts, rather like 4 white Cuisenaire rods in a row.] If I use...

[I. interrupts himself, making sure:] You wouldn't have any trouble at all showing somebody five quarters with money, right?

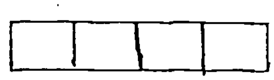
62. H. Um-hm [agrees; she could do it easily]

63. I. Suppose we tried to represent quarters...[still drawing as he talks] I can get quarters better than I can get fifths, because I can divide it [the rectangle] into half, and then I can divide the halves into half.

So... there!... I've got quarters. [displaying the paper]

How could you show me five quarters?

[Remark: The paper shows



Of course, at this point you CANNOT show 5 quarters. Something must be added!]

64. H. You could add another quarter...

[65. Remark: This is half of the crucial truth here. You MUST have more. The other half of the crucial truth is the realization that making 5 pieces won't help -- you will just have 5 fifths, instead of 4 fourths. In the process of "getting more", you must not

change the UNIT. But the interviewer didn't dare test to see if H. had this second part of the story figured out. It involves a meta-analysis that H. probably had never carried through.

So -- instead of probing for H.'s understanding of the role of the "unit", I. chose to state his questions in a way that he hoped would skirt the issue, and avoid confusion.]

66. I. O.K.

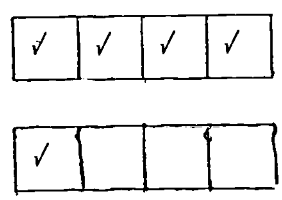
Indeed.

And, in fact, what people often do it to imagine another whole thing down here...

[draws a second rectangle, intended to be congruent to the previous one]

...and then divide that into fourths...

[I. now places check marks in 5 of the "quarters"



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and say 'There's one, two, three,four, five'! That's really what it is!

67. H. Um-hm [She appears to accept this as a solution.]

68. I. Now... somebody could come along, and look at what we have here, and say "You've got one, two, three, four, five, six, seven, eight... So! Why aren't those eighths?"

That's sort of a good question.

Let's try it another way. [I. counts out 8 25-cent pieces ("quarters").] There are eight of these coins -- so why don't I call them "eighths"?

69. H. [Happily, H. sees the same pattern in both the rectangles and in the money!] Because, in these groups right here, there's only four!

70. I. Nifty! That's nifty! [So -- H. has at least the rudiments of the concept of "unit"!]

That's just right.

I want to repeat what you just said. You said "Because in this group right here there's only four." [gestures of H. and I. both indicate the first rectangle and the first dollar's worth of coins.]

What's the key thing?

What is it that we really divided up, here, to get those fourths, when we called those "quarters"?

What did we divide up?

71. H. A dollar

72. I. [inflection indicates agreement] a dollar ... and [gesturing to the next stack of 4 quarters] this was a different dollar that we divided up.

And so when we were dealing with these rectangles, we went on to another rectangle, a different rectangle...

That's exactly the answer.

That's really very nice.

[73. Remark: Now that the foundation of the concept seems secure, I. decides to attach the word "unit" to it.]

74. I. Do you know the word "unit"?

A word mathematicians sometimes use...

I'll write it.

[Writes: "unit"]

75. H. I've heard of it... but never really used it.

76. I. O.K.

People would sometimes say: "We use the dollar as the unit here."

I've forgotten the exact phrase you used a moment ago -- you had a very nice phrase for it. Something like: "the thing we divided up." Or maybe you said "This group right here." Whatever...

What you called "this group right here" is what mathematicians would call "the unit". So, if we take the dollar as the unit, then these coins ought to be called quarters.

77. I. Suppose, now, we wanted to talk about pizzas.

[Interrupts himself, once again making sure of closer contact with the student's ideas:]

Can you draw a pizza?

78. H. Without the topping?

79. I. [Laughs] Yes. I think so, yes!

80. H. [draws a circle]

81. I. O.K. And now, if you wanted to show people five fourths with that, what would you do?

What is it we're using as the unit?

82. H. The pizza.

83. I. Yeah. Whole pizzas.

So... if you wanted to show people five fourths, how many units would you want?

84. H. Two. [very quickly and confidently; she knows!]

85. I. Two. Exactly right!

I don't think you're very confused about that at all! I think you've been very clear about it.

Do you think of yourself as good at mathematics?

86. H. Yah! [confident and happy]

87. I. Yeah. I think you're very good at mathematics.

88. H. I used to be pretty horrible at it. I'm pretty good at it now. [Once again, perhaps, an allusion to this deficient school that H. used to attend -- wherever that was. The interviewer preferred not to ask.]

89. I. What made the difference?

90. H. I don't know.

[So... maybe H. was NOT alluding to "that other school"...]

[pause]

[91. I. brings out some Cuisenaire rods.]

92. I. Have you ever worked with these Cuisenaire rods?

93. H. A few times. We did some project in September with them. [Note: This interview took place in June.]

[94. I. turns to a new page on the pad of paper.]

95. I. I want to...

[Interrupts himself:] There are many different things you can do with these [referring, of course, to the Cuisenaire rods].

95.(cont.) ...but suppose I decided that I wanted to call this red rod "one".

96. H. All right

97. I. Can you decide what name I should give to some other rod, there? Find any other rod, and tell me what you'd call it.

98. H. That's "one"! [Pointing to the red rod; her inflection indicates that she is reasonably sure she understands what's going on -- up to a point, anyhow -- but just wants to make assurance doubly sure.]

99. I. Yup. That's "one". [The red rod is being called "one".]

100. H. This is probably "a half".

[indicates white rod, which is 1 cm. long; the red rod, of course, is 2 cm. long.]

101. I. Exactly right! And so that the people listening to the [audio] tape will know, which color did you pick up?

102. H. Beige...brown...sort of...

[In fact, it was what most people call one of the "white" rods, but H. is right. It certainly isn't really white.]

103. I. [Laughs] You are really very accurate, do you know that?

O.K.

Can you find a rod in there that you might call "two"?

[I. spills out some more Cuisenaire rods onto the tabletop, to make them more accessible.]

104. I. [observing what H. is doing]

O.K.

You're looking for some kind of rod that you might call "two" ... and [saying it for the sake of the audio-tape record] you're trying the yellow...

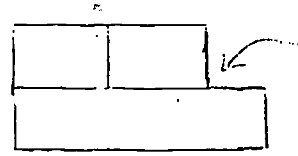
105. H. No...It doesn't work.

106. I. ...and are you going to call that "two"? [H is holding a yellow rod, which is 5 cm. long -- Note that H. probably does NOT know the dimensions of these rods, in centimeters. Nor would I. use them in the interview. The dimensions are reported here <u>only for the sake of adult readers of this report who may not know the lengths of the various rods</u>.]

107. H. No

108. I. Why not?

109. H. It's not exactly two. [H. has seen the intention of using <u>length</u> as the decisive attribute, and has performed the crucial test: two reds, placed end-to-end alongside a yellow rod, do not make a "train" the same length as the yellow rod:



1 cm. gap

.]

110. I. You're right.

111. H. It's not exactly two. [H. repeats her previous remark; she's thinking about the situation.]

112. I. What <u>would</u> you call it, incidentally? [This is a somewhat harder question, but I. wants to make sure he isn't grossly underestimating H.'s level of knowledge.]

113. H. The yellow rod?

114. I. Yeah...

If you were calling the red rod "one", what <u>would</u> you call the yellow rod?

115. H. Oh... [pause]... [uncertainly]...about one and a half... <u>TWO</u> and a half! [She quickly corrects her error.]

116. I. Yes!

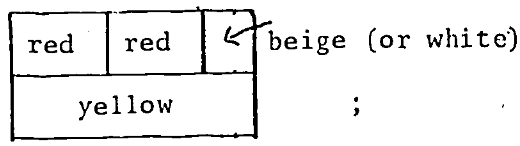
Can you describe what you've got there, for the sake of people who hear this tape...because they can't see...?

117. H. O.K.

I've got a yellow rod. Then I've got two red rods...

118. I. ...and a white rod -- or, as you say, beige...

[What H. has put together looks like this:



;

since she has recognized that the chosen criterion is to be the length of the rods (or of "trains" of rods), and that the beige rod has length one-half (i.e., it is half as long as the red rod, which has been chosen as the unit), her arrangement of rods does, in fact, prove that the yellow rod "should be called 'two-and-a-half' " (i.e., has length 2.5, with the red rod as the unit).]

119. I. I'm probably going to keep calling that [the 1-cm. rod] 'white', but I agree with you that 'beige' or 'brown' would be more accurate. You have to be a little careful, because most people call this one [holding up the 8-cm. rod] "the brown rod".

Can you see any rods there that look like they ought to be called "two"? [The stress on the word "look" was intended to suggest the two-step process that H. was, in fact, using: first, make a visual estimate; second, make a precise array of rods that proves the correctness of your estimate.]

120. H. Probably this...

121. I. O.K. What color would you call that?

122. H. Ahmm...purple

123. I. Good. That's what I'd call it, too. Purple.

124. I. What would you call the orange rod? [In this transcript, the language "would you call" can sound artifical and confusing; in fact, H. interprets it correctly from the very beginning. Most other students do, also. This "giving names to the rods" approach is very reliable. We have used it with -- literally -- thousands of students over the past 20 years.]

125. H. This one? [holds up an orange rod]

126. I. Yes.

Can you guess what we ought to call that rod?

127. H. Oh... about...[hesitation]... five.

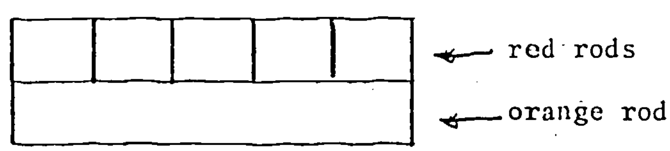
[128. Remark: her visual guess is, of course, exactly correct!]

129. I. O.K. [said with a non-judgmental inflection]. Do you want to try it, and see if it works? [This brings us to the second task: make an array of rods to prove that 5 is correct.]

130. H. Um-hmm [agrees; and starts to do so.]

131. H. Yeah. Five. [Happy at the confirmation]

[132. Remark: Her array, of course looks like this:



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133. I. Exactly right!

Well, now... you say that you didn't used to think you were good at math, and now you think you are... I think you're VERY good at math...

What do you think made the difference?

134. H. Probably my teacher...

135. I. I guess that's often the answer ...

O.K.

I am having trouble finding any question you don't already know the answer to. [This is not accurate; H. is doing an excellent job at working out answers, but she clearly did NOT "know" them a priori -- i.e., she did NOT know beforehand that the orange rod was as long as a train of 5 red rods, etc. The strength she is demonstrating is strength in resourcefulness, not strength in prior factual knowledge.]

136. I. Let's try something else.

Suppose I call the light green rod "one". [He holds one up.] What would you call the dark green rod? [He holds up a dark green rod.]

137. H. That one? [making sure they're talking about the same rod]

138. I. Yeah...

139. H. Yeh, it looks pretty green [she has been checking out the color, and now agrees]

140. H. It's "two." The dark green is two.

141. I. Exactly what I would call it.

Can you find a rod you'd call "three"? Which one do you suppose it would be?

[142. Remark: The interviewer's strategy here was determined as follows:

i) He wants to move on to consider the problem

$$\frac{1}{3} + \frac{1}{2} = \quad .$$

ii) Doing so will require a new choice of unit. Calling the red rod "one" provides for halves, but cannot, of course, provide for thirds.

iii) Bruner and others have reported that, when a single choice has always been made [here, if only the red rod has been used as a unit], then people tend to be unaware of the possibility of making any other choices; but as soon as a person has seen two possibilities,

he or she easily moves on to imagine many more possibilities.

iv) Hence, I. introduces the possibility of taking the light green rod as a unit.

v) Since H. will have seen two different possibilities -- the unit as the red rod, then taking the unit to be the light green rod -- she should be able to deal with other choices of the unit.

vi) It is NOT accidental that I. has not included the correct choice of unit for the problem

$$\frac{1}{3} + \frac{1}{2} = \ldots \quad .$$

Only by requiring students to produce an original, "unprecedented" answer (or "non-imitative" answer) can I. be sure that the student has used the concepts and not merely imitation.

vii) In terms of the real-time interview, I. started (in the first sentence of #136) to move on to the problem

$$\frac{1}{3} + \frac{1}{2} \quad .$$

Then he suddenly realized that this would violate Bruner's Law: H. had seen only a single choice of unit, and might have trouble moving on to different choices, which she would need to do in order to solve the problem. Hence, I. interrupted himself, as he very often does.]

143. H. The brown one.

144. I. O.K. [non-judgmental inflection]. Let's try it.

145. H. ah... n-n-no [thoughtfully]. No.

[laughs]

The blue one! [happily]

146. silence [I. does not like to respond too quickly, and also tries to avoid any evident pattern -- so here he waits a moment in silence .]

147. I. That's exactly right!

O.K. What would you call the brown rod? [Which, as a matter of fact, is actually 8 cm. long.]

148. H. The brown one? [H. always like to check out the question before she works on the answer.]

149. I. Yup.

150. H. One-and-a-half ... maybe...

No! Two-and-a-half!

151. [pause]

152. I. Well, let's

153. H. Red would be "a half", wouldn't it, if the green is "one"?

154. I. Well... maybe...[doubtful inflection]

If the green is "one", would the red be "a half"?

155. H. ...[brief thoughtful pause]... No.

156. I. No [he agrees] I think you're right.

158. I. What would you call the "beige" or "white" rod? What would you call that, if the light green rod is "one"?

158. H. A Third?

159. I. You're right.

How would you settle that, if someone said he didn't understand why you were calling that "a third"? What would you do, to show him why you were calling the beige rod "one third"?

160. H. [immediately] I'd probably take one of the green rods [she means light green], and stack 3 of the beige rods alongside the green rod... and show him that it just fits.-..

161. I. That's exactly right!

Well... I have trouble finding questions to ask you... you're good at all of these things... You're so good at figuring it out, which is what math is really all about...

Let me try quite a hard problem.

Suppose I wanted to talk about <u>fifths</u>. Which rod would I call "one," so that I'd be able to talk about <u>fifths</u>?

[162. Note that here, also, I. is leading up to the $\frac{1}{3} + \frac{1}{2}$ problem, where H. will have to make her own choice of unit, and unless she makes this choice correctly, she will not be able to solve the problem. Hence, I. wants her to get ready by thinking about how one does choose an appropriate unit.]

163. H. ...[pause]... Maybe... The yellow one.

164. I. You're exactly right!

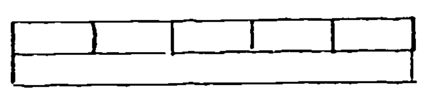
How would you convince me that, if I called the yellow rod one, I'd be able to talk about <u>fifths</u>?

What would you do, to show me?

165. H. Ahmm.....Take some of these...

161. I. Those are the famous beige-brown-white rods...

167. H. Yeah ... and kind of ... put 'em next to each other, up against the yellow.

[]

168. I. That's exactly right!

All right...

...and if you wanted to show somebody "one fifth plus two fifths", what rod would you call "one fifth"?

169. H. The beige one.

170. I. What rod would you call "two fifths"?

171. H. [no hesitation] the red

172. I. What rod would you call "three fifths"?

173. H. The green [she picks up a <u>light</u> green rod]

174. I. O.K.

175. I. I want to show you what most people think is really a hard problem, namely,

[writes: $\frac{1}{2} + \frac{1}{3} =$]

one-half plus one-third ...

Do you know what that is?

One half plus one-third

176. H. ...N-N-No... No.

177. I. Why do you suppose people think this is a hard problem? You just did "one fif plus two fifths is three fifths". Now what makes "one half plus one third" harder than "one fifth plus two fifths"?

178. H. Ahm... Because one-half is larger...?

179. I. ... That's true, ... [i.e., one half is larger than one third -- but this, of course, is not the source of the difficulty, as

$$\frac{3}{7} + \frac{1}{7}$$

demonstrates.]

180. I. Let's come back to that...

That's the problem I really care about... but let's go on, for a moment, to a different problem.

Suppose I had one half plus one fourth

[writes:

$$\frac{1}{2} + \frac{1}{4}$$],

what would that be?

181. H. Ahmm... one third. ...Ah... Two thirds!

182. I. O.K. [non-judgmental inflection]

Do you want to write that?

[I. was quite surprised by H 's answer; he suspects something interesting -- and something that he doesn't understand! -- is going on here . Hence, he wants to slow down the action, and

get as much data down on paper (and on tape) as possible.]

What color (pen) do you want?

183. H. Ah... blue

[I. gives her a blue pen.]

184. H. So... what do I write?

185. I. Whatever you think the answer is.

One half plus one fourth equals... whatever you think...

186. H. Hmm...[pause]

Maybe...[pause]

No... [pause]

Maybe... two... thirds... or ... [reflectively; she is uncertain...]

[pause]

187. I. Well, you put down [on paper] anything you think the answer is, and then we'll see if we can figure it out with the rods.

[188. Remark. H's tone of voice makes this transition even more dramatically apparent. From the start of the interview, through the first 170 or so utterances [as numbered here], H. has been happy, confident, sometimes thoughtful, always resourceful, occasionally wrong -- but not often. And when she has been wrong, she has always easily corrected her error.

But now, beginning with #180, her tone of voice suggests that she is lost. She seems totally adrift, unable to find anything to hang on to.

This is the first big transition in this interview session, a session that can be divided into three parts:

Part I. H. is dealing with matters for which she can create adequate mental representations; she handles everything with resourcefulness and confidence.

Part II. We move into "paper-and-pencil" arithmetic

for which she is unprepared. (Although students her age are ordinarily expected to know this content, it is clear that H. does not.) Specifically, we encounter the problems

$$\frac{1}{2} + \frac{1}{3} \quad =$$

and

$$\frac{1}{2} + \frac{1}{4} \quad = \quad .$$

H. is no longer resourceful. She becomes unable to investigate, to set sub-goals, etc.

Part III. [which will come later].

When the same problems from Part II were tackled, in Part III, as concrete questions about concrete materials, H. could bring to bear all the resourcefulness she had demonstrated in Part I. and could easily solve these problems.

It has long been suspected that school programs could be created that could build on students' "concrete" or "experiential" or "informal" knowledge, and use this as a foundation for building up a powerful student capability for dealing with "formal" mathematics.

This has been difficult to demonstrate at the level of school programs, in part because the creation of potentially effective programs is no small task. But in this interview, at the level of ONE student at ONE MOMENT in her life, the possibility emerges clearly. If one builds on concrete experiential knowledge, H. deals with these problems creatively and powerfully.

189. H. I guess it's...

[pause]

I'm not sure...

[pause]

190. I. Well, let's try the rods...

If we want to talk about "one half" and "one fourth", which rod do you suppose we want to call one?

191. H. [pause]

Ahmm...

[pause]

Light green?

192. Silence]

193. I. That's a good guess...

...but I think that might not be our best bet...

194. [silence]

195. I. Remember, when we call the light green rod "one", we get thirds, OK?

196. H. Um-hm. [inflection indicates agreement]

197. I. ...I don't think you're going to get fourths or halves [if you call the light green rod "one"].

198. H. Oh, yes. [Her tone of voice indicates that her self-assurance and resourcefulness are rapidly returning.]

It's probably the magenta or purple. [The "purple" or "lavender" or "magenta" rod is 4 cm. long, and is the optimal choice for this problem.]

199. I. Yeah. Exactly right.

That's just what we should do.

Now... so we can keep track, I'm going to stand a purple rod on end, here, where we can both see it. That should remind us that the purple rod is "one".

O.K. If that purple rod is "one", which rod would you call "one half"?

200. H. The red

201. I. Exactly correct!

Yeah.

That's exactly right.

And now... which rod will I call "one fourth"?

202. H. The beige

203. I. The beige. Good. Exactly right.

O.K. So, what happens if you add one half and one fourth. What will you get?

204. H. Three fourths [Note: at this point, H. does not need to use the rods! They have served their purpose -- we would say they have enabled her to build up an appropriate representation in her mind; she can now solve the problem easily just by thinking about it. Because she now has an adequate mental representation, she is able to think about it!]

205. I. You're right!

So... let's not erase that answer [her previous answer,

$$\frac{1}{2} + \frac{1}{4} = \frac{2}{3} \text{]},$$

because I think that's interesting, but here we have "one half plus one fourth"

[writes: $\frac{1}{2} + \frac{1}{4} = \_\_$,

below the preceding incorrect result],

and... what do you get now?

206. H. Three fourths

207. I. ...you want to write that answer...?

[208. She does.]

209. I. That's really a very nice job!

O.K.

Let's come back to the really hard problem.

I don't think it will be hard for you, but a lot of people find it extremely hard.

[writes: $\frac{1}{2} + \frac{1}{3} =$]

If you want to talk about "one half" and "one third" -- well I need to be able to find one half of some rod, and I need to be able to find one third of it...

...and so... which rod will we call "one"?

210. H. Ahmm... the dark green?

[Of course, she has chosen correctly!]

211. I. I think you're probably right. [Actually, I. knows she is right, but he doesn't want to remove the need to check it out.]

You really are very good with those.

Let's try it, and check up and see...

If dark green is "one," which rod is "one half"?

212. H. Maybe purple... No... Ahm... Green! Light green!

213. I. Light green! Exactly right!

So we know, then, that light green is "one half"...

... and which rod would you call "one third:?

214. H. [instant response] The red!

215. I. You're right!

O.K.

So what, now, will "one half plus one third" be?

216. H. Ah...

[pause]

...one third is...

I don't know...

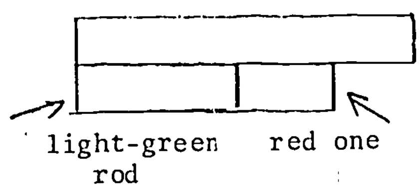
...four fifths?

217. I. Can you do something with the rods so that you can show me a rod that represents the answer? We'll worry about what to call that rod later on... Let's find the rod, first...

Show me a rod that's the right size...

218. [pause]

[219. H. puts down a dark green rod, then a light green rod alongside it, then she adds on, end-to-end, a red rod:



]

220. I. So... how are you going to show me "one half plus one third"? Just exactly what you're doing!

Will you describe what you're done? [So as to get it on the audio tape...]

221. H. I just took the dark green and put the light green -- "one half" -- right next to it , and then the red, as a third...

[pause]

222. I. That's exactly right!

[pause]

223. I. We need a name for these. Some people call these "trains", because they look a little like a train, with cars in a row...

So , you made a train of a light green and a red, right next to the dark green rod, there,...

... and that's certainly the right answer. That is the right size, to be "one half plus one third"

...and now we need to figure out what to call that...

[pause]

What do you suppose we should call it?

224. H. Ahmm...

I don't know...

225. I. [very softly] Well...try...

226. [pause]

227. I. How could you decide?

228. H. I'm trying to think of something... I just can't...

229. I. O.K.

Now you're picking up one of the rods. Which rod are you picking up? [I. wants to get a record on the tape.]

230. H. Ah... beige...

231. [pause]

232. H. ... I don't know...

233. H. ... probably six or seven [she is estimating how many beige (or white) rods would fit alongside the dark green rod]

234. I. O.K. What will we call the white rod, then? The beige one, that is?

235. H. One seventh. [Her estimate is wrong, but her method is entirely correct. If it were true that seven beige rods would make a train as long as one dark green rod, then it would also be true that the beige rod would represent one seventh. But, of course, her visual estimate is wrong. The fact is that six, not seven, white rods make a train as long as one dark green rod.]

236. I. [I. does NOT want H., or any student, to be looking to the teacher (or the interviewer) for the determination of "truth". The strength of mathematics and science is that they are NOT entirely authoritarian; one attempts to determine the truth directly,

wherever possible.]

How would you settle it?

Let's do something to be sure...

237. [silence]

238. [H. actually lines up a train of white rods alongside the dark green rod.]

239. H. It's six! [Happy to have it settled.]

240. I. Exactly correct!

So... we call the beige rod...?

241. H. One sixth.

242. I. ...and what are you going to call the train that you made up? With the light green and the red?

243. H. Hmm... Three sixths?

[244. H.'s visual estimation is surprisingly poor... Many students do much better.]

245. I. You want to check up and make sure you got it right?

[246. silence]

247. H. Hmm... No... Five sixths!

[Again, she sounds pleased to get the matter settled.]

248. I. That's exactly right!

O.K.

Let me write that...

[He writes

$$\frac{1}{2} + \frac{1}{3} =$$

on a new line on the paper.] I'm writing "one half plus one third equals...", ... and will you write what you now think that is equal to?

249. H. writes:

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

250. I. That's really a nifty job!

That's very nice.

You're really very good at mathematics.

You say you just started studying fractions?

251. H. Yeah. I didn't have very much [about fractions] last year. I've just come to <u>this</u> school. [Before that] <u>we</u> didn't <u>have</u> very much fractions...

252. I. Where were you before you came here?

253. H. In Michigan. Near Pontiac and Detroit.

254. I. O.K. You've been a big help to me. [The students understand that these interviews are part of a research program.]

If everybody could deal with math the way you can, it'd be wonderful! [This was a sincere remark. Notice that H. has been able to fight her way through to a correct answer in each key problem! She did not come to this session already knowing how to add fractions, but she has used her determination and her very considerable resourcefulness to work through the problems, and to arrive at correct answers.]

What do you suppose I should say to other people in order to help them to get to be better at mathematics?

255. H. [pause]... I don't know...

[This may partly be modesty...]

256. I. What I think <u>you</u> do, that is really <u>terribly</u> valuable, is that you really <u>think</u> about the problem.

257. H. Oh... I wouldn't know what to <u>say</u> to them... [i.e., to anyone who needed help] ... I wouldn't know what to <u>tell</u> them...

258. I. It seems to me that when you are working on a math problem, you really think about it. A lot of other people want to be able to solve problems <u>without</u> really <u>thinking</u> about them... That

doesn't usually work... You <u>have</u> to think about it...

Do you have any questions <u>you</u> want to ask <u>me</u>?

259. H. No... not really... I think I understand it.

Friday, January 7, 1983

Kenwood School -- Kay Andert

Stefan -- grade 1

1. I. What grade are you in?

2. S. First

First grade...m m

3. I. What you could help me do would be: how could we explain the idea of one-half to somebody. OK?

Can you imagine somebody who didn't know what half of a thing is? Now...how would we help them learn what half of a thing is? We could use these Cuisenaire rods, if that would help.

Do you know how to write "one half"? Have you even seen it written? [pause] Have you ever heard about "one-half"?

4. S. No

5. I. [repeating] No... Never... Suppose I said: Do you think you could find a rod that was half as long as that [I. holds up, then lays in front of S., the dark green rod, which is 6 cm. long.] What would you do for that?

Could you find a rod that was half as long as that one?

6. S. This here? [points to the dark green rod]

7. I. Yes.

Stefan, with no hesitation, immediately makes the correct choice and picks up a light green rod [3 cm. long].

8. S. This one? [The light green rod]

9. I. That's right.

Now, suppose somebody said "Gee, I don't know what you mean when you say that that light green rod is half as long as that dark green rod!"

Could you do anything to give them an idea of what you mean by that?

10. S Ahmm...

11. I. I think you could maybe do something... ... do something with another rod...?

12. S. picks up a second light green rod and puts three rods together, showing that two light green rods, placed end to end, are just as long as one dark green rod. [Cf. Figure 1].

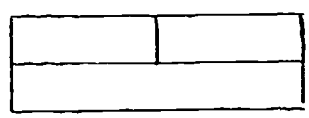


Figure 1

13. S. Yeah!

14. I. ...you could say, "Both of them [the light green rods] are the same length, and two of them together are just as long as the dark green rod, so one light green rod by itself must be <u>half</u> as long as the dark green rod."

Suppose we tried something else.

Suppose ... Let's get out some of these red rods. [The interviewer has decided to switch from <u>length</u> and a <u>continuous variable</u> to deal, instead, with the <u>number</u> of identical rods (he chooses red rods, each of which is 2 cm. long), and with a <u>counting situation</u> involving <u>discrete</u> collections.]

How many would you take?

17. S. I would take two. [S. says this immediatley and confidently, obviously pleased that he can answer so quickly.]

18. I. Yeah...that would be exactly right!

You didn't have to think very long to do that! You knew that right away!

19. S. ...cause I knew two plus two is four [again, happy and confident]

20. I. Unh-huh... and so...[pause in talking while I. gets some more red rods out of the box] Let's see...that's one, two, three, four,... seven... Is that eight? [i.e., "Do we now have 8 red rods on the table in front of us?"]

21. S. Unh-huh [intonation implies a confident "Yes"]

22. I. O.K. Suppose now you were going to take one-half...

How many, now, would you take?

23. S. Ahh... [pause of about 3 seconds]

Four

24. I. You're right. You're certainly right.

Why don't you take four,...and I'll take four...

[They do so.]

Now we can pile them up and see if we actually got the same number. We can see if your stack is the same height as my stack.

[They do so, and the stacks match.]

So... we each got half.

[pause; then sound of rods clinking against one another as I. searches through the box of rods, looking for some more red rods.]

You're very good at this, you know.

How many red ones have we got here, altogether?

25. S. Ten

26. I. How did you figure that out? [Stefan has not done any obvious direct counting of the rods.]

27. S. Well, because I knew that four plus four is eight, and [then] I counted "nine, ten" [with his eyes, not his fingers].

28. I. Perfect!

Suppose you were going to take half of all of them. How many would you take?

29. S. I would have ... five! [again sounding happy and confident]

30. I. You <u>are</u> very good at that! You're really very good!

Oh, wow!

...[pause of about one second]

Well, if you were sharing with <u>three</u> people, then you'd say you were taking <u>thirds</u>. Let's see if we have any more red rods in here [i.e., in the box; he finds two more]

How many red rods have we got <u>now</u>?

31. S. I don't know ... [pause]...Counting <u>all</u> these? [gestures to the various piles of red rods on the table]

32. I. <u>All</u> those...

33. S. Eight

34. I. I think it's more than that...

35. S. Every single one of them [i.e., of the red rods on the table]

36. I. Yeah ...

37. S. Twelve

38. I. Yeah, twelve...

How did you do it?

39. S. Well, first I counted eight from these... then I went "nine, ten, eleven, twelve."

40. I. Now, instead of just you and me, suppose we were going to share these among three people... Suppose <u>you</u> were going to get some,

and I was going to get some, and maybe we had some third person... [I. decides he needs a more concrete piece of imagery, so he interrupts himself.]

Who in the class would be somebody else we might share with? Just give me a name...

41. S. Kelly

42. I. Kelly? All right. You're going to get some, and I'm going to get some, and Kelly is going to get some. We want to be fair, and all get the same number.

How many do you suppose we'd each get?

43. S. ...each get four! [answered very quickly, happily, and confidently.]

44. I. How did you figure that out?

45. S. Well, I knew that we'd each had four, and then I counted four... [Notice that this does NOT fully explain how S. thought about this problem. He is NOT referring to the immediately preceding problem, which dealt with one-half of ten being five. Possibly S. used some kind of "estimate, then count each (imagined) pile to see if the division was fair. He is NOT actually touching or moving the wooden rods, here. Indeed, probably neither he nor the Interviewer could have solved the problem so quickly if they had attempted to move the actual rods around on the table. S.'s answer came very quickly!]

46. I. You are very good at that!

People would often say, because there were three of us -- you, me, and Kelly -- because there were three of us, we each got one third.

And so we could say: "one third of twelve is four."

47. S. Un-huh [agrees; presumably he has never heard this language before, or at least never attended to it...]

48. I. Let me show you how you'd <u>write</u> some of those.

[He writes:

$$\frac{1}{2} \text{ of } 8 = \quad ,$$

and reads it aloud.]

I could say "one half" -- people write "one half" like this [as he writes the

$$\frac{1}{2} \quad]$$

...have you ever seen that before?

49. S. Yeah [apparently he has]

50. I. ...we could write "one-half of eight is..."

That's one we did just a minute ago...Do you remember what that is? How much is one-half of eight?

51. S. No.

[This is in some ways a surprising result. A few minutes earlier S. had worked this out, and was quite confident about it. We would analyze this as an instance of "verb" behavior -- something you can <u>do</u>, given a flow of externally-supplied feedback (such as seeing the actual rods), but have not yet transformed into <u>noun knowledge</u> -- the process is not yet a sequence of actions, welded together cognitively, so that you can <u>think about</u> the process <u>without</u> actually doing it.]

52. I. You want to think about it for a few minutes and see if you can figure it out? One half of eight? See if you can figure that out...

[pause of about 3 seconds]

53. I. What do you think?

54. S. No

55. I. Well, what could we... If we wanted to make that into a problem, ... [Note that S.'s attention is now mainly on the written notation $\frac{1}{2}$ of 8 =, on the paper, and not on the rods.]

What that says is, we had eight things, and you and I wanted to share them equally, so that we'd each get one-half.

How many would you get?

56. S. I'd get two. [Again, confident -- but this time he's wrong.]

57. I. Well, there are eight ... if we each take one half...

58. S. Oh! I get it! Six. [confident, but wrong]

59. I. Six [repeating S.'s answer with what I. hopes is a non-judgemental inflection]

60. S. Yeah! Four, five, six!

61. I. ...but we've got eight altogether. [S.'s attention, for utterances 48 through 61, has been on the written problem on the paper, and not on the rods.]

62. I. This number here is now many we've got altogether [indicates the '8' in

$$\frac{1}{2} \text{ of } 8 =].$$

63. S. Oh! Hmm... Ahhh...

64. I. If we share them equally, there are two of us, so we'd say we each get one half.

65. S. Seventeen

[At first, this choice seems to defy explanation. But wait!].

66. I. How did you figure that out?

67. S. Well... First I went 'eight', and then real quickly I counted 'nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen'.

[S. has taken the '8' as one person's share, and computed what the original total must have been ... but in his 'real quick' counting,

he has made an error, and gotten 17 when his method should have led to the answer "sixteen".]

68. I. ...Well... Let me come back to this later. [I. feels there is now too deep an accumulation of confusion surrounding this problem, so he chooses to go on to a fresh start on a new problem.]

Another one! Let's try some other numbers. [He also focuses his attention on the rods, rather than on the paper.]

Suppose we have...

How many of those red rods have I got there, now? [I.e., on the table.]

69. S. Six

70. I. O.K. Suppose we shared those fairly, so you got the same number I got.

How many would you get?

71. S. I'd get...three! [He actually starts to share out the red rods, but his imagination jumps ahead of his fingers, and he proclaims the answer, without completing the "sharing-out" distribution.]

72. I. Yeah. And people would say that you got half, and I got half, because there were two of us. [Note: In utterances 69-72, I. has revised the sequence, compared to utterances 48-67. In 48-67, the temporal sequence starts with a notation on paper, and hopes to elicit the construction of a playing-out of the sharing process with wooden rods; by contrast, the sequence 69-72 starts with a sharing of physical rods, and then uses records written on paper as a means of recording what was actually done.]

73. I. And so we could write that and say [writing as he talks] "One half of six is...--" let's see: the whole number was six, and you got half of that... How many did you get?

74. S. ...six? ...

75. I. That's right, the whole number of all of the rods we started with was six, and you got half of that... How many did you get?

76. S. ...ah...three [reasonably confident]

77. I. So... we say "One half of six is three!" [writing it as he says it]. Do you see how that works?

78. S. Yeah. [sounds reasonably happy about it]

79. I. Suppose we had four to start with [he assembles 4 red rods in a pile on the table] ... What would one half of four be? [pause of 7 seconds]

80. S. Eight.

[Here he makes exactly the same error he made earlier (65-67) -- he has retrieved an immature frame that has three place-holders, that we might call A, A, and 2A. However, both now, and in utterances 65-67, he takes an input datum that should match with the 2A slot, and matches it with one of the other slots. In the present case, he consequently gets 4, 4, and 8. The next two utterances serve to clarify somewhat the method he is using.]

81. I. How did you do it?

82. S. Well... I knew that four plus four was eight! [triumphant!]

83. I. Oh... But suppose four is all there are, and you're going to take half, and I'm going to take half...

84. S. How many would I take away?

85. I. Yeah

86. S. Two [very confident, and pleased that he understands]

87. I. Yeah... and that's what we would say that was.

[writes: $\frac{1}{2}$ of 4 = 2]

[Comment: The Interviewer worked to show S. that his first frame-retrieval-and-mapping (utterances 80-82) had NOT succeeded, and I. tried to use the experience of "sharing out equal portions" to begin building a more mature frame in S.'s mind. At least in

the short run, I. seems to have been successful.]

88. I. [Reviewing what has just been done with rods, and recorded on paper...] Let's see; we've done... for half of six, we got three, and for half of four we got two...

Let's think about this "half of eight" problem [returning to the first problem on the paper, but now writing it over, to preserve the time-sequence order of the lines on the paper]

89. S. [sighs]

90. I. What could we do? To think about this "half of eight" we need to make up a story about sharing ... if we want to match "one half of eight," then how many red rods should we start with?

91. [pause of about 6 seconds]

92. I. Altogether we should have eight, right?

You count out enough to have eight.

[S. does so.]

Now, if you wanted to find out how much half of eight is, you could share them equally with me, and what you'd get would be half, and what I'd get would be half.

[I. is being worried that he has not yet made contact with a foundation of things that S. knows well, so I. interrupts himself to seek a firmer foundation in things that S. already knows...]

Do you have any brothers or sisters?

93. S. Yeah. I have one sister.

94. I. Have you ever shared a candy bar with her?

95. S. Yeah.

96. I. How do you do it?

97. S. Well... I split it... and then I give one half to her...

[pause]

98. I. ...and keep the other half for yourself...

99. I. O.K. Suppose we wanted to do that with these [the 8 red rods in a pile on the table]. These might be marbles, or something... and you are supposed to get half of them, and I am supposed to get half of them...

How many would you take? [pause for about 6 seconds]

100. S. Four

101. I. You're right!

Can you write that... say "one-half of eight would be...". How many?

102. S. Four

103. [I. writes

$$\text{"}\frac{1}{2}\text{ of } 8 = \quad\text{"}$$

and S. finishes it, by writing "4".]

104. I. What do you suppose one-half of twelve would be? That's really a hard one...

105. S. That's a hard! [S. uses the adjective as a noun]

106. I. [agreeing] That's a hard one!

107. S. [sounding as if he's guessing] Eleven?

108. I. How did you do it?

109. S. Well...because...I knew that it was one, and then I added twelve more... [Clearly something has malfunctioned.] ...and I counted them...up...

110. I. I see ...

111. I. What'll we do? We wanted twelve to start with, right?

Can you write "one half"?

One half of twelve?

112. S. S. writes

$$\frac{1}{2}$$

113. I. "...of twelve..." See, if we each get half, then how many we get depends on how many we start with, right?

114. I. Maybe we can figure it out...

First of all we need 12 [I. counts out some more red rods]

O.K.

Now, how much is half of 12?

115. S. Eleven?

116. I. Well, what will we do with these pieces of wood?

117. S. Ahh...

118. I. If you and I took the same amount, then you'd get half... Isn't that right?

119. S. Yeah!

120. I. So...let's do that!

121. S. How many I would get?

122. I. Yeah...

123. S. Ungh! [or some such sound]

[pause of about 6 seconds]

124. S. I would get...Seven?

125. I. Let's do it, and see!

[They share out the red rods.]

126. S. [really triumphant] Six!

127. I. [agreeing] Six [This is plainly apparent, since the rods have been shared out.]

So... we can say "One half of twelve is..."

128. S. Six!

129. I. You're right!

That's exactly how it works!

Let's see... what have we done? We've done 4, and said half of four is two... We've done 4, and 6, and 8...

Oh! Ten! We haven't tried ten. What do you suppose half of ten would be? How many would you have, if you had half of ten?

130. I. Can you figure it out, before we do it?

131. S. Six?

132. I. Well... let's try it, and see...

133. S. I mean... ungh...

[They share out the ten red rods, equally.]

134. S. Five [entirely confident]

135. I. Five. You're right! Why don't we write that one?

One half of ten would be five.

[S. writes. Actually, in both of the last two equations he has omitted the word "of". I. sees possible trouble stemming from this, but decides this is not the time to bring the matter up.]

136. I. That's exactly right!

Friday, June 4, 1982

Todd Stoltey - Grade 5

Kay Andert's class

[Some discussion of tape recorder, which color of pen to use, etc.]

1. I. Do you have any younger brothers or sisters?

2. T. Yeah

3. I. How old are they?

4. T. My sister is ... ah...eight!

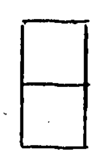
5. I. O.K.

Well, she might already be too old, conceivably ... but can you imagine somebody who's younger than you are, and doesn't know about fractions?

How would you explain "one half" to them!

[I. writes: $\frac{1}{2}$.]

6. T. You can probably make a...make a... [While he is talking, he draws



.]

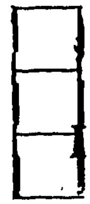
box, and then draw a line right in the middle.

7. I. O.K. And you've just done it, there, haven't you, on the paper! That's just what I would do. How would you explain two-thirds

[writes: $\frac{2}{3}$

to somebody? If they didn't know about fractions?

8. T. I would make another box [draws rectangle] and then split it up [by now he has drawn



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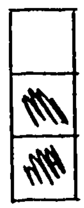
and then color in two...ah...two [he can't seem to find a word, such as "pieces" or "parts," to describe the subdivisions he has just drawn.]

[By now he has drawn



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9. I. That's nifty! Yeah! That's nifty! You didn't have any trouble with either of those, did you!

How would you explain <u>five-fourths</u> to them?

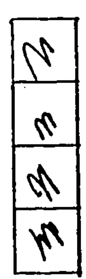
[writes: $\frac{5}{4}$.]

10. T. [pause]

Hmm...

[pause]

[Without talking, T. draws on the paper:



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11. I. Now that's just what <u>I</u> would draw!

Can you explain, for the sake of the tape, what it is that you

have drawn on the paper?

12. [brief pause]

13. I. ...You drew a rectangle, to start with, right?

14. T. I drew a rectangle, then I drew three lines to split it up, evenly.

15. I. ...so that's a rectangle divided into fourths, isn't it?

16. T. I colored in four. [He did.]

17. I. O.K. You've just showed me four fourths. How can you show me five fourths?

18. [pause]

19. I. What I think would be a very good idea -- why don't you draw another rectangle, the same size as that one...

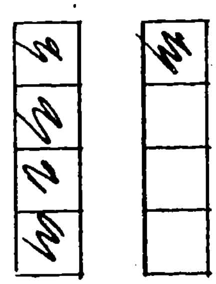
20. [T. does so.]

21. I. ...and let's divide it into fourths

22. [T. does so.]

23. I. ...and now, how many are you going to color in?

24. T. One...on this box... [referring to the second rectangle; the picture now looks like this:



]

and these ... [gesturing to the four sections shaded in on the original rectangle] ... five fourths!

[His tone of voice shows he is pleased. He has shown five fourths!]

25. I. Yeah! Sure is! Absolutely perfect!

O.K.

A lot of people tell me there couldn't be 'five fourths.' What do you suppose they say? How many fourths would you guess is the most they say there could be?

26. T. ...probably... They probably think...say...four...and...because split it up into four, and they wouldn't have no more boxes [rectangles, or parts of rectangles].

27. I. Yeah, that's right.

And, what I've been saying to people when they tell me that -- when they say "There couldn't be five fourths" -- goes like this.

[He writes on the paper:

$\frac{1}{4}$.]

There's another name for "one fourth" that people use. What else is it called?

28. [silence]

29. I. What else is "one fourth" sometimes called?

30. [silence]

31. I. Well, when people tell me there couldn't be more than four fourths, I usually show them some of these.

[He puts three 25-cent coins on the table.]

What are these called?

32. T. Quarters

33. I. Why are they called "quarters"?

34. T. There's four quarters in a dollar, and one of them...there's...

35. I. Yeah. And that's another name people often use for "one fourth", isn't it? People often say "one quarter."

I guess "one fourth" is its official mathematical name.

Now, when people tell me there can't be more than four fourths -- four quarters -- I show them something like that. [He puts some more quarters on the table, making a total of six.]

How many quarters have we got there?

36. T. Six

37. I. Right.

Could there be more than six?

38. [By facial expression, T. indicates "of course".]

39. I. Of course there could.

In fact there's seven [putting down one more], and there's eight [putting an eighth quarter down on the table].

Do you agree?

40. [T. indicates that he agrees.]

41. I. If I've got <u>eight</u> coins there, why don't I call them "eighths"?

They're all equal...

And I've got eight pieces...

Why don't I call them "eighths"?

Instead of "quarters"?

42. T. You're not... you're not using eight.

43. [pause]

44. I. Well... I put <u>down</u> eight...

45. [pause]

46. T. ...'cause you need...you need to make another box]i.e., rectangle; note that, despite the present conversation concerning <u>money</u>, T. is mapping data into his reliable assimilation paradigm, based on <u>drawing rectangles!</u>]

...and you need to mark fourths [in this second rectangle], and you need to take five fourths of them, and you don't use those. [T. is referring to the 3 sections of the second rectangle which he had not shaded in, in Utterance 24 (or so), when he was "showing someone five fourths". Note that his logic here is seriously in error. Whether these coins should be called "fourths" or "eighths" or whatever does NOT depend upon <u>how many there are</u>; it does NOT depend on "having some left over" (as he is now claiming). It

DOES depend ONLY on how the unit is divided up. In this example with coins, the unit, of course, is the dollar. Since the dollar is divided into four equal parts, each part is "one fourth". If T's logic were applied to his earlier (correct) handling of "two thirds", one would have to conclude that those fractional parts should NOT be called "thirds", because we didn't use all of them.]

47. [pause]

48. I. ...Oh...Kay...[thoughtfully; I. is unsure of how to interact with T.'s incorrect conceptualization]

If I were talking to somebody about eight fourths [One standard interview technique used frequently in this study is to accept, momentarily, the student's "rules of the game," and try to explore what those rules would lead to. I. uses this strategy here. By considering eight fourths, I. sweeps away any considerations of "having some left over". Such considerations were always irrelevent; instead of arguing about their relevance, however, I. concocts a problem where they do not appear at all.] ...suppose I were talking to somebody about eight fourths [I. repeats himself], how many of them would I use now?

49. T. All of them?

50. I. All of the ones that we've got on the pad, there.

And...now I'm really using eight. Why don't I call them "eighths" instead of "fourths"?

51. [brief pause]

52. I. There are certainly eight of them... [The question, #50, still hands in the air.]

53. T. Because you're talking about fourths, and you need these eighths [sic! So! T. is, in fact, prepared to talk about

"eighths" here!] to get the fourths that you need.

[Probably T.'s basic idea is mostly correct here, but his language fails him -- and, of course, he has NOT identified the key role that is played by the unit!]

54. I. Everything you say is OK...when you said "we were talking about fourths", that's right... But why, even though I've got eight equal things, do I continue to talk about "fourths"? Why are we really talking about fourths, even though we've got eight equal things?

55. T. 'Cause if we write the problem right [Here T. is beginning to home in on the correct idea -- but it's ONLY a beginning. Obviously, the next question should be: "What do you mean by 'if we write the problem right'? What determines what is 'right'?"] ...There's eight fourths...there's four of 'em here [he points to th first rectangle -- now he's really getting close to the truth! -- one might pick up on this and ask: "What do you mean by 'here'? What is special about what you're calling 'here'?" Answer: That (original rectangle, or that dollar) is our unit. The interviewer does NOT interrupt immediately to pursue these points, of course, because he wants to see how far T. will get on his own!] ...and you have four more [pointing to second rectangle] 's eight and that's eight fourths... [T. has not addressed the real question!

56. I. ...ah...I don't know whether you would have met this word...It's a word mathematicians use sometimes... The word "unit". Some people would say: "Well, O.K....the thing that really was the unit...the key thing that we really divided into fourths, was what? [Note that I. is not aiming primarily at whether T. knew the word "unit", but rather at whether T. had realized that there is a key thing that we agree to focus our attention on. This

"key thing" is then divided into n equal parts, each of which is therefore called "one n-th" . In any given problem, one may have any number of these "n-ths", just as a piggy bank could contain a hundred or more "quarters" -- so-called because the dollar has been divided into four equal parts, no matter how many we have in the piggy bank.

We emphasize this, because the present study has been concerned primarily with ideas, and NOT with the words in which those ideas are expressed. A student who did not know the word "unit" could express the key idea quite satisfactorily in other words -- for example, by saying: "In any discussion of counting or measuring, you must make a decision to focus on some particular kind of thing; if you were dealing with people, you'd have to decide whether to count individuals or families or classrooms of students or whatever. After that decision -- say you picked classrooms -- then you could talk about "half of a class", which would mean the class was divided into two equal parts. In a large school, you could easily have 20 or more halves of classes. But that wouldn't make them "twentieths", because each of the classes would be divided into only two equal parts, so each part would still be one half of a class."

This would express the key idea reasonably well -- and it does not use the word "unit".

Many typical school programs pay too much attention to words, and too little attention to ideas.]

[57. Recall that I. asked:
"...the thing we really divided into fourths was what?"]

58. T. Quarters? [sounds unsure of himself]

59. I. Well, that's what we got... but the thing we divided into four equal pieces, in order to get those quarters, was what?

60. T. Eight? [sounds unsure of himself]

[61. pause]

62. I. Tell me again why one of these [25 cent pieces] is called "a quarter"?

63. T. ...'cause there's four quarters in a dollar.

64. I. O.K. So <u>what</u> is it that I divided into four equal parts?

65. T. One whole.

66. I. One whole <u>what</u>?

67. T. One whole <u>dollar</u>. [sounds pleased]

68. I. Yeah, exactly right!

So, people would say that the dollar was the unit, is that right?

[In the following statement, I. is not really looking for information, but rather seeks to emphasize, for T.'s personal benefit, the importance of analyzing one's own actions and one's own thought processes:]

Now, in what you did... Everything you did was correct. I'm not really arguing with you. I just want to find out <u>why</u> you say those things.

What did you use as the "unit" up here [points to pictures of two rectangles on the paper], where you were talking about the five fourths? You drew something that was just right. What did you use as the "unit"?

69. T. That box [which is what he calls rectangles].

70. I. Yeah. That box. [agrees]

71. T. [interrupting; he wants to re-state his answer] That <u>rectangle.</u>

72. I. That rectangle [agreeint].

<u>Exactly</u> right! I think you see very well how that works!

[pause]

You're really pretty good at that.

Do you think of yourself as good at mathematics? Or not good at mathematics? Or what?

73. T. I'm usually good at <u>fractions</u>. [He sounds pleased.]

74. I. Oh. [sounds appreciative.]

How long have you been studying fractions?

75. T. Since second grade.

76. I. Oh, wow! O.K. Then I believe you probably <u>are</u> good at fractions.

77. I. [getting out some Cuisenaire rods]: I want to to give names to these rods, here.

Suppose I want to call the red rod "one"... What name would you give to some <u>other</u> rod, if I want to call the red rod "one". [I. prefers to give questions in the <u>least-structured form</u> possible -- hence he does NOT direct T. towards any specific rod.]

[78. T. picks up a light green rod]

79. I. [describing the action for the sake of the audio tape] O.K. You picked up a light green rod. What will we call that?

80. T. Two? [There is a question suggested by his inflection]

81. I. O.K. How would we decide whether that was really the best name to give it, or not?

82. T. [drops light green rod (on floor, in fact), and picks up a purple rod.] No [rejecting the light green]; this one [i.e., the purple]. This is "two".

83. I. O.K. You've picked up a purple rod. And you propose to call that "two." How can we decide if that is right, or not?

84. T. [showing with the actual rods]...because two of thse red ones equals one of those [purple rods].

85. I. O.K. You made a little tower there, with one red rod on top of another, and it was just the same height as the purple rod. Very clever!

I need a <u>name</u> for rods put end to end like this. Let's call it a train, because it sort of looks like a train, with the freight cards all lined up there. O.K.?

86. T. [indicates "O.K."]

87. I. O.K. So the red rod is one. I'll leave a red rod standing up on the recorder here, to remind us that "red is 'one'" .
And the purple rod is "two."
Which rod would you call "three"?

[88. T. immediately picks up the dark green rod]

89. I. ...and you picked up, instantly, which one?

90. T. The green

91. I. Yeah. The dark green.
And now how would you convince somebody that you really had the right name for that?

92. T. [puts a purple rod next to the dark green rod, apparently hoping to use visual estimation]

93. I. ...Can you describe what you've done, so we'll have it on the tape?

94. T. I picked up a green [dark green], to see if this purple one'd be more than half of it...

95. I. O.K. Now, I think you're <u>right</u> in calling the dark green rod "three". [The interviewer has to express agreement to T. much more often than usual, because T. is very excitable, and is quick to assume that he's being challenged, as by a silence, a careful glance from I., or by most questions.]
But suppose someone was <u>still</u> not convinced. What could you do, to convince them? [pause]

96. I. Which rod are we calling "one"?

97. T. The little red one...

98. I. What are we calling "two"?

99. T. The purple

100. I. ...and you say the dark green should be called "three".

If somebody wasn't convinced, what could you do, to convince them?

101. [pause]

102. I. I mean, you <u>are</u> right, but somebody might not believe it...

103. T. I could take these red rods... [does so]

104. I. That is really nifty!

Want to describe, for the tape, what you did?

105. T. Took the green rod [the dark green rod], picked up three little red ones, layed them on it to see if they'd fit...

[i.e., he made a train of 3 red rods <u>on top of</u> a horizontal dark green rod.]

106. I. That is just nifty!

That certainly ought to settle it. Anybody who didn't think it was three now would be just plain dumb, wouldn't they?

107. I. O.K.

Can you find a rod you'd call "four"?

Can you do your stunt of picking it up right away without...

<u>You did</u>! Good!

Which color did you pick up?

108. T. Brown.

109. I. Yeah.

And if somebody doubted you, what would you do?

110. T. Hmm... Let's see.

Take the brown rod. Take two purple ones... [pause]

111. I. ...and what is the purple rod known as?

112. T. Two

113. I. [interpreting T.'s construction]...and so you've got 'two plus two equals four', right? So the brown one should be four. You're certainly right!

114. I. Ah...can you find a rod that should be called "five"? Can you do the same stunt, picking it up immediately without touching any other rod?

You did!

...oh, no, you didn't...

I actually found a problem you couldn't solve <u>instantly</u>...

115. T. ...I think...<u>this</u> one, right?

116. I. That's right. You're right!

The one that you picked up a moment ago, that wouldn't work, was the blue one, right?

But the orange one would work.

What could you do to show someone that the orange rod really ought to be called "five"?

117. T. Take the orange rod, and five little red ones, and...[he makes a "train" of 5 red rods alongside one orange rod:

orange				
red	red	red	red	red

].

118. I. Suppose I wanted to be able to talk about <u>thirds</u>. Maybe I wanted to show somebody what "one third" is, and "two thirds", and so on... Which rod should I call "one"?

119. T. One? [Mostly, T. is just repeating the interviewer's words while he thinks about it.]

120. [T. picks up the purple rod (which in actual fact is 4 cm. long, and is therefore unsuitable.).]

121. I. Which one did you pick up?

122. T. The purple one.

123. I. [recapitulating] O. K. -- if you call the purple rod "one", then we ought to be able to talk about <u>thirds</u>, is that right? Which rod

would you call "one third"?

124. T. One-third?

125. I. One-third.

[126. T. picks up the white rod.]

127. I. What color rod is that?

128. T. Tan

129. I. O.K. The tan rod. Most people call that the white rod, but you're right. "Tan" is a better name for it. May I call it "white"?

130. T. Yeah .

[131. T. tries to lay three white (or "tan") rods alongside the purple rod, and is surprised at the result.]

132. I. ...and we've just discovered something. What did we just discover?

133. T. That it's not the right one. [I.e., that the purple rod is not the correct choice for "one" if you want to deal with thirds.]

134. I. You're right!

[135. By gestures, T. chooses the red rod to be called one.]

136. I. You want to call the red rod "one"?

137. T. Yeah.

138. I. Than which rod will be "one third?"

139. T. mmmm...

140. I. Now I'm getting mixed up. What are we calling the red rod?

141. T. One.

142. I. ...and the one you have in your hand is...?

143. T. One-third. [sounds confident]

144. I. And what color is it? [What color is the rod in your hand?]

145. T. Green

146. I. [for clarification] That's the <u>light</u> green, isn't it? Now suppose you wanted to convince somebody that that light green rod should be called "one-third"?

147. T. Take three of these little white blocks [1 cm. long], and put 'em... [as he talks, T. puts three white rods alongside the light green rod] ...there's three of them, so that's three...

[148. Note that T. has reversed the roles of the white and light green rods. He has, in fact, demonstrated that, if the light green rod is called "one", then the white rod should be called "one-third." But a moment ago T. wanted to call the red rod [2 cm.] "one", and to call the light green rod "one third."

If you assume intact structures, there seem to be three different structures lurking in this discussion:

Structure A:

red rod $\longleftrightarrow$ one

white rod $\longleftrightarrow$ one half

(and there are no thirds!)

Structure B:

light green rod $\longleftrightarrow$ one

white rod $\longleftrightarrow$ one third

(but T. does not get this mapping set up correctly)

Structure C:

white rod $\longleftrightarrow$ one

light green rod $\longleftrightarrow$ three

(This is a correct isomorphism, preserving addition, but it does NOT lead to fractional names.)

It appears to be the case that, after abandoning the idea of calling the purple rod "one," T. attempted to set up the mapping for Structure A. He seems to have perceived quickly that this was not leading him to thirds, and to realize that, if purple was too long, and red too short, he should probably try light green. This is a correct perception, and T. begins to move toward Structure B.(at least to the extent of focusing on the light green rod, and using the name

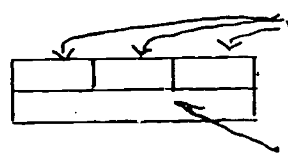
"one-third"). However -- and this is particularly interesting -- he somehow gets the mapping reversed:

white "one-third"
light green "one"

At about this same time -- soon thereafter, if his spoken words are a correct indication -- Structure C intrudes, he obtains an isomorphism that preserves length, <u>but he loses the whole idea of fraction in the process</u>!

It is easy to see the competition between these different structures and different mappings as a kind of "spreading activation" phenomenon, perhaps in the sense of Minsky's K-lines [Minsky, 1980]; that is to say, <u>logic</u> may not lead from one to the next, but <u>similarity</u> and <u>pattern recognition</u> do, at least to the extent of making the "competition" among these alternatives seem reasonable.]

149. I. [Responding to T.'s array:



O.K. I agree with what you just <u>did</u>... Ahmm... that's certainly right. You put the three white rods alongside the light green rod, and they just fit exactly right. That's certainly what you mean by "thirds." But I'm getting confused about what <u>name</u> we're giving to which rods.

[I.'s purpose here, of course, is to get T. to talk about the <u>structures</u> and the <u>mappings</u> so as to make his choices more clearly recognizable.]

What will you call the light green rod?

150. T. One-third. [T. is being consistent. This is what he called the light green rod back in utterance number 143.]

151. I. O.K. You want to call it "one third" [I.'s tone is intended to convey neither approval nor disapproval, but to be merely repeating T.'s choice in order to guarantee effective communication between them, much as the waiter in a restaurant might repeat your order.]

152. I. Ahmm...what are you going to call the white rod now?

153. T. One half? [T.'s inflection makes this a question. The answer itself is totally unexpected, and takes I. by surprise!]

154. I. O.K. <u>You're</u> telling <u>me</u> [I.'s inflection says: "You're doing it. Just tell me what you want!"]

One half.

O.K.

What are you going to call the <u>red</u> rod?

155. T. One(?) [T.'s inflection makes this <u>almost</u> a question; he is not sure...]

156. I. I think we'd better start writing all of this down! I'm getting all mixed up! [Note that the mapping T. has suggested is as follows:

white $\longleftrightarrow$ one half

red rod $\longleftrightarrow$ one

light green rod $\longleftrightarrow$ one third

The first two correspondences on this list are consistent with one another, but the third cannot be reconciled with the two above. Given the kinds of evidence from other sources that are listed in Davis, 1983, it is tempting to suggest that the underlying structure C (described above) -- which is surely best known to T., since it is basically simple counting, "one, two. three,..." -- is the main knowledge representation structure that he has in mind, that is shaping his naming of the light green rod. As in Davis, 1983, the underlying structure may be <u>internally consistent</u>, yet the <u>words</u> that are used in input/output operations may be wrong. T.

seems to be confusing "one third" with "three."]

157. I. Can you write down... I guess there are several ways of doing it. Would you rather write down the names, or would you rather trace around the rods themselves?

158. T. Write down their names.

159. I. O.K. Would you write down the name for this rod [the white rod]. Call it "t-a-n" if you like.

[160. Here is the table T. wrote down:

tan	$\frac{1}{2}$
l. green	$\frac{1}{3}$
red	1
yellow	2]

[161. The last line in the table was written in response to this question from the interviewer, after T. had written:

red 1 :

I. Have you got a rod you'd call "two"?

T. picked up a yellow rod, and then added the bottom line to his table.]

162. I. O.K. Suppose somebody came along, and didn't like the way we were naming these.

Let's pretend you're a lawyer, and somebody comes along and wants to hire you to prove that <u>this</u> system of names [the table in utterance 160] doesn't work.

What could you do? You want to prove that this system of names won't work...

163. T. You mean... they could use "yellow" as "two and a half" [T. is rather confident that he has said something worthwhile.] [This, taken together with earlier remarks, is fairly conclusive evidence that T. <u>is</u> aware of <u>lengths</u>, and is trying to construct mappings

that preserve length (or "size") isomorphically. The fact is that, if red is one, then the yellow rod should be named "two and a half"! Only this name preserves additivity.]

164. I. Now that's really nifty!

[165. Without speaking, T. builds an array of five white rods alongside one yellow rod.]

166. I. O.K. So you're saying the yellow rod should really be called "two and one half", right?

167. T. Um-hmm [means "yes"[

168. I. If the yellow rod is "two and a half," can you find a rod that should be called "five"?

[169. no response from T.]

170. I. Suppose that that "two and a half" were money -- dollars -- how much would it be? What's another way of saying "two and a half dollars"?

171. T. Two dollars and fifty cents.

172. I. Exactly right. If you had two "two dollars and fifty cents's", how much would you have?

173. T. Five dollars.

174. I. Exactly right.

So, if the yellow rod is "two and a half", which rod should be called "five"?

175. T. The orange.

176. I. Which rod would you call "two"?

177. T. [Picks up the purple rod and waves it at the interviewer.]

178. I. How would you convince somebody that that [the purple rod] should be called "two"?

179. T. Take... Take two red rods and lay 'em on top of the purple rod...

180. I. O.K. Just what I would do!

181. I. Which rod would you call "one"?

182. T. The red...

183. I. What would you call the white rod?

184. T. That's still a half.

185. I. ...and...what would you call the light green one?

186. T. [immediately and confidently] One and a half.

187. I. Exactly right! Nifty!

188. I. So... I guess you'd win that case!

[189. The new table, which they have written down, looks like this:

tan (white	$\frac{1}{2}$
l. green.	$1\frac{1}{2}$
red	1
yellow	$2\frac{1}{2}$
orange	5
purple	2]

190. I. O.K. That works very nicely, if you're talking about halves -- but suppose we want to talk about thirds?
Now we want to talk about thirds. Which rod will you call "one"?
We know if you call the red rod "one", that gives you halves, not thirds.
So which rod will I call "one"?

191. T. Purple.

192. I. [pause] ...Ahmm... I think if you try that you'll find that it won't work. That'll give you quarters ... that'll give you fourths... but it won't give you thirds... [Recall that T. has, in fact, tried this choice earlier in the present session! I. thought it best not to remind T. of this fact. (Cf. utterances 118-128.)]

193. T. Hmm... Then it should be...[pause]... the green.

194. I. I think you're right! Let's call the light green "one." Do you want to write that? The light green we'll call "one."

[195. Here T. begins the construction of a new table, by writing

light green 1 .]

196. I. O.K. We're calling the light green rod "one." Which rod would you call "two"?

197. T. Yellow.

198. I. ...and if somebody doubted that, how would you convince them?

199. [Note that T.'s error in utterance No. 197 is probably NOT a serious one, but more likely merely an error in his estimation of length. T.'s skill in estimating length is less than one expects in students his age.]

200. T. Take out two greens [he means "light green rods"] and lay them on top of...Oop! [When he tried it, it didn't work!]... Hmm... It should be this. [He picks up a dark green fod.]

201. I. For the sake of the tape recording, which color did you pick up?

202. T. The dark green.

203. I. Exactly right! Want to record that?

[204. T. now extends the table, so that it reads:

Light green	1
Dark green	2]

205. I. ...and which rod should we call "one third," now?

206. T. The red? [His inflection makes it a question.]

207. I. How would you show somebody that the red rod should be called "one-third"?

[208. Note that I. tries to avoid rendering judgments of "right" or "wrong". We advocate this, both in teaching situations and in interviewing for data collection. There are several reasons for this, including these:

(i) We want students to believe, in the words of Jerrold Zacharias, that "science [including mathematics] is a game played against nature; it is never a game played against the teacher." It doesn't matter what the teacher thinks; the truth or falsity of a mathematical statement should be something that is determined by reality. (Admittedly, the "reality" of mathematics is an abstract reality -- here, the criterion is whether the mapping is an isomorphism that preserves the additivity of length.)

(ii) We want students to become self-reliant in such matters.

209. T. Take three of them and ... [T. completes the idea, not in words, but in actions: he attempts to lay 3 red rods alongside a "green" rod -- but. whereas this would be the correct definition of "one-third" if he were using the light green rod, in fact T. is using the dark green rod. (Recall that T. frequently says "green" when clarity would really require a qualifier.)]

210. I. Would you say, for the sake of the tape recording, what you have built there?

211. T. Ah... I took a green rod, and I put three red rods over the green.

212. I. Did it fit?

213. T. Yeah.

214. I. Yeah, it fitted exactly right! Which "green" are you using?

215. T. The dark green.

216. I. O.K. ... and what are you calling that dark green rod?

217. T. Two

218. I. And so it looks to me like you've got one-third of two, there... But when you want to call a rod "one third", you want the rod that's one-third of one...

219. T. Then <u>this</u>!

220. I. You're right! [T. has put three white (or tan) rods on top of a light green rod.] Which one is one-third?

221. T. The tan.

222. I. Right! Why don't you write that?

[223. T. writes a third line in his new table, which now reads:

Light green	1
Dark green	2
Tan	$\frac{1}{3}$

224. I. We just have time for one more. Are you ready for a hard one?

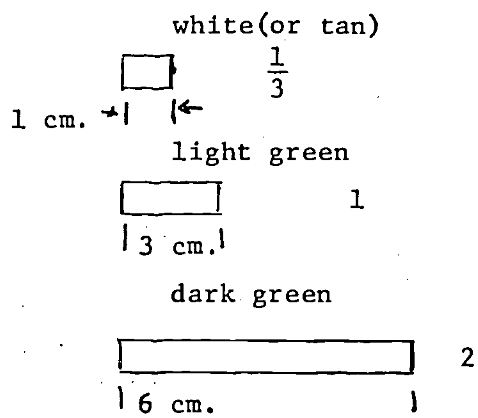
[225. T. agrees, by a nod.]

226. I. If that's all true -- with <u>this</u> system of names -- light green is one, dark green is two, tan is one third -- with this system of names, what would you call that black rod? [The black rod, in actual fact, is 7 cm. long.]

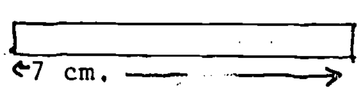
227. T. Four

228. I. How would you convince somebody?

[229. Comment: T.'s answer of "four" in utterance No. 227 is, in its way, stunning! How on earth can T. claim that? The naming system that the interviewer and T. have, together, built up looks like this:



and here is the 7 cm. black rod:



How can this be seen as "4", which would be the correct number name for a rod 12 cm. long (and therefore equal to "2 + 2," or 6 cm. plus 6 cm.), if there were a rod 12 cm. long (in this set there is not)? From another point of view, the fact that the black rod is as long as a dark green rod and a white rod together almost cries out to be noticed, which would lend to the correct name for black -- specifically, $2\frac{1}{3}$. T. has made use of the additive property of length earlier in this interview. Why not now?

This seems to follow a pattern that we have seen in other interviews (cf., e.g., R. B. Davis, "Frame-Based Knowledge of Mathematics: Infinite Series", Journal of Mathematical Behavior, vol. 3, no. 2 (Summer, 1982), pp. 99-120). In all such cases, our interpretation of the data is that the students have failed to synthesize adequately-structured representation structures. Somehow they are trying to operate with sketchy, incomplete, inadequate structures -- rather as one might try to interpret history with only the two categories of "good guys" and "bad guys"; so deficient a structure would not allow you to represent history satisfactorily, and these students cannot represent mathematical ideas satisfactorily.]

[230. T. lays down a black rod, and tries to put four reds on top of it. They don't fit -- but whether they fit or not, notice that this would be the wrong criterion, anyhow. For this correspondence, it is the light green rod that plays the role of a unit (i.e., "is called 'one' "). T. ought to have been putting light green rods on top of the black rod, if he wished to use this method of deciding.]

231. T. ...four of these... and lay them on top... [T. is talking as he carries out the action of item 230, above.]

232. I. Would you describe what you've got there?

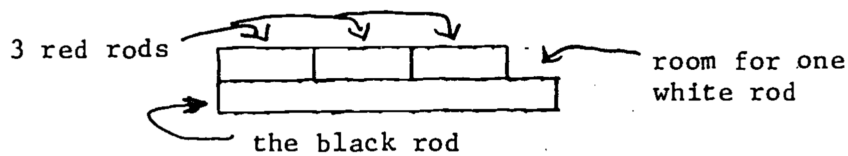
233. T. They don't fit.

234. I. I agree it doesn't fit, but for the sake of the tape, would you describe what you've got there?

235. T. The black rod on the bottom, and four of the red rods on top.

236. I. Right. You were able to make those red rods stick there, but they don't really fit, they don't have the right length.

[237. T. puts three red rods on top of a black rod, and shows recognition that there's just enough space left for a white rod



238. I. Ah. What have you got now? [The Interviewer wants T. to describe in his own words the configuration he built in No. 237, above.]

239. T. Three and one half -- three and one third! [T. changes his answer immediately.]

[240. At this point, T.'s difficulty becomes clearer. In building up a mental representation for this mathematical situation, T. has used the idea that "red is one", and he is working with a larger representation that includes this faulty component part. Most of his previously mysterious behavior -- e.g., calling the black rod "four" in utterance number 227 -- becomes perfectly intelligible, once we know that T. is working with the idea that "red is one." Of course, if T. performed routine checks on the adequacy of his representation, he would have discovered that "red is one" is incompatible with the table of correspondences that T. himself has written on the paper.]

241. I. Ah...Now we need to be careful...

Say what you've got, will you, so we can be sure to get it on the tape.

242. T. The black on the bottom, and three red rods [on top], and there's space left for one of the tan rods [on top].

243. I. Right.

O.K.

And now, what does that tell us about what we should call the black rod [i.e., "what number name should we give it?"]?

244. T. Three and one third?

245. I. That's very clever, but...

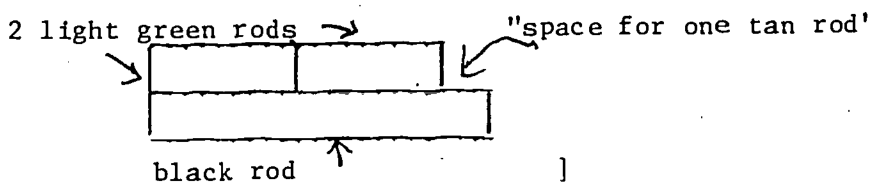
246. T. No! It's not one! [T. has suddenly recognized his error in using "red is one" as an ingredient in his representation.]

247. I. You're right! EXACTLY right!

The red rod is NOT "one." The little tan rod is one third.

That was right!

[248. While I. was speaking (in utterance no. 247), T. was removing the three red rods, and replacing them by two light green rods:



]

249. I. What are you doing now? You took off what you had a moment ago -- and what have you put there now?

250. T. Two greens. [T. sounds happy!]

251. I. Yeah. Two of the light greens.

252. T. [Sounding very pleased about it all.] It's two and one third!

253. I. That's exactly right! Why don't you write that down?

259. T. completes his table, which now reads:

light green	1
dark green	2
tan	1/3
black	2 1/3 .

255. I. Thank you very much.

You _are_ good at fractions.

Did you have any questions you want to ask me?

[256. T. indicates that he does not, and leaves the interview area to return to his regular classroom. He was quite evidently feeling happy.]

ERD contains $\frac{13}{5}$ mapping error May 28, 1982

1. I. Let me write it...

[writes:

$\frac{1}{2}$]

Suppose you were trying to explain to somebody what "one half" meant -- how would you explain it?

2. E. You take a .... a whole, and ... divide it in ... in two parts.

3. I. O.K. How would you explain what "two thirds" meant?

[writes

$\frac{2}{3}$]

4. E. I would take something and make ... make three piles... and take... two of them...

5. I. O.K. That's just what I would do.

How would you explain something like "thirteen fifths" to somebody, if they didn't know about it?

[writes: $\frac{13}{5}$.]

6. E. ...Well...I...That's...that's a sticker!

7. I. Yeah, it is.

[8. pause -- silence on the tape]

[9. Comment: Given our postulated model, we interpret this data as a likely attempt by E. to retrieve the same representation structure (or "frame") that she used in utterances 2 and 4 -- which explains

$\frac{a}{b}$

by taking something, dividing it into b parts, then taking a of these parts. She then tries to map "5" into "total number of parts", and "13" into "number of parts to be taken". But of course this mapping fails so self-evidently that the failure cannot be ignored. This failed mapping of input data into slots in the knowledge representation

structure leads E. into the confusion that she reveals in utterance No. 6.]

10. E. That's supposed to be thirteen divided by five, or just "thirteen fifths"?

[11. Comment: E. is the first subject interviewed to bring up, on her own, this important distinction. We interpret this as indicating that E. does more evaluating of the correctness of a representation, and more comparisons among possible alternative representations. Life experience may explain part of this. E. is in her eighties, and for fifty-five years she worked as a legal secretary and tax specialist. Now retired, she was the first adult to be interviewed in our study.]

12. I. Well... it comes out to be the same thing, actually. [This was probably an inadvisable remark for I. to make; conceptually the two are very different, indeed -- or may be, depending upon the definitions you use.]

[13. pause -- silence on tape]

14. E. I don't know how I would do that.

[15. brief pause]

16. E. I would make thirteen piles,...
...and take five of them
...and see how many I had left.
But that isn't quite right, either. I don't know.

[17. Comment: At least four interesting things have happened here:

(i) First, E. reverses the mapping of input data into frame slots, so that, instead of mapping

"5" $\longrightarrow$ "total no. of parts"

"13" $\longrightarrow$ "number of these parts that you take"

--which, of course, fails [but is fundamentally the correct pattern, provided one introduces the concept of unit] -- she uses the mapping

"13" → "total no. of parts"

"5" → "the number of these parts that you take"

(ii) For some reason -- presumably as the result of an internal evaluation -- she rejects the step of completing the a/b representation, which would now call for $\frac{5}{13}$.

Presumably something about

$$\frac{5}{13}$$

alerts E. to the existence of some sort of difficulty, contradiction, or error.

(iii) E. now switches to a <u>different</u> knowledge representation structure (or "frame"). Specifically, she switches to the representation structure for <u>subtraction</u> ("...and see how many I had left.")

(iv) True to her typical behavior, E. -- unlike our elementary school students -- carries out <u>another</u> internal evaulation, and says, "But that isn't quite right, either."

For E., this is what one learns to expect. For most of our subjects, this would be highly unusual.]

18. I. Well... O.K....

19. E. [interupting] How would you divide that, really? [She is genuinely curious.]

20. I. Well... You'd do just the same thing you did up here [gestures to where he has written "$\frac{1}{2}$" and "$\frac{2}{3}$" on the paper].
You'd take something -- whatever -- a <u>pie</u>, or whatever it is [he draws a circle] -- and divide it into five pieces -- it would be like that -- now, I'll need <u>several</u> pies [draws two more circles (or "pies"), for a total of three]... ...and I divide each of them into five equal pieces [draws this, more or less] -- let's see, I've got ten pieces there, so I need the third pie [draws dividing lines in the "third pie"]... now, I can take 5, 10, 11, 12, [coloring them in], thirteen

pieces, and each piece is one fifth -- one fifth of a pizza, or pie, or whatever this is -- O.K. Is that all right?

21. E. Oh. [inflection suggests partial comprehension ... nning to dawn]

22. I. ...and that's really just what you did up here [gestures to "$\frac{1}{2}$" and "$\frac{2}{3}$", as previously written on the paper]. You took a unit, and divided it up. You took a unit, and divided it up into three parts.

23. E. Mmm. [inflection suggests growing degree of comprehension]

24. I. I try to get students to look first at the denominator, and you did that up here. You took something and divided it into two pieces [for $\frac{1}{2}$]. You took something and divided it into three pieces [for $\frac{1}{3}$]. So, for

$$\frac{13}{5},$$

you take something and divide it up into 5 pieces. Well, that's not going to be enough to give you 13 pieces. So, then -- if this is, say, pizza -- you need to have another pizza. That still won't do it. That'll give you 10 pieces. Each of those parts is a fifth, all right, because you're dividing each pizza into 5 equal pieces -- a fifth of a pizza -- but that only gives you ten, so you need to get a third pizza, and then you can take thirteen. So that does it!

25. E. Um-hm. [Reasonably satisfied]

[26. Remark: E. shows a pattern which most people, including experts, probably use often. If one mapping of input data into frame-slots fails, switch some things around to get a new mapping, and see if that will work. It resembles the way one might insert plugs into sockets in connecting up electrical equipment -- or the way one might reverse a key if, on the first try, it won't fit into a keyhole. This is probably a very valuable procedure.